Chapter 1 Exercises

Exercise 1.

The following expressions are sentential functions: (a), (d), (e), (h). The rest ((b), (c), (f), (g)) are designatory.

Exercise 2.

For an example of a sentential function from geometry, the Pythagorean Theorem works: $a^2 + b^2 = c^2$. For an example of a designatory function, we can use the formula for the area of a circle: πr^2.

Exercise 3.

(a) Category (ii), since no number can have $x = x + 3$

(b) Category (iii), since this equation is satisfied by 7 and -7.

(c) Category (i). Expanding $(y + 2)(y - 2) < y^2$ we get $y^2 - 4 < y^2$ which is true for all $y \in \mathbb{R}$.

(d) Category (iii), since this equation is satisfied by any $y \geq 13$.

(e) Category (i), since this is true for all $z \in \mathbb{R}$.

(f) Category (ii), since this would require $z > z + 12$, which is never true.

Exercise 4.

For an example of a universal theorem from arithmetic, we can use the associative property of addition: $\forall x, y \in \mathbb{Z}\ x + y = y + x$.

For an absolutely existential theorem, consider defining a simple property: $\exists x \in \mathbb{Z}$ such that $x + 1 > 2$. Then this is true for (amongst many other numbers...) $x = 2$.

For a conditionally existential theorem, we can define another simple property: $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $x + y = 10$. Then, given any choice of $x \in \mathbb{Z}$, define $y = 10 - x$. Then $x + (10 - x) \Rightarrow x + 10 + (-x) \Rightarrow x + (-x) + 10 = 10$.

Exercise 5.

(i) $\forall x, y\ x > y$. This is false, since $x = 1, y = 2$ is a counterexample.

(ii) $\exists x, y$ such that $x > y$. This is true, since letting $x = 2, y = 1$ satisfies the statement.

(iii) $\forall x, \exists y$ such that $x > y$. This is true as well, since for any $x \in \mathbb{Z}$, let $y = x + 1$.

(iv) $\exists x$ such that $\forall y$, $x > y$. This is false, since this is to say that there is a biggest x in $\mathbb{Z}$. For any x, $y = x + 1$ works as a counterexample.

(v) $\forall y$, $\exists x$ such that $x > y$. This is true. For any y, pick $x = y + 1$.

(vi) $\exists y$ such that $\forall x$, $x > y$. This is false, since this would be to say that there is a smallest y in $\mathbb{Z}$. For any y, $x = y - 1$ works as a counterexample.

Exercise 6.

For the first sentential function:

$$x + y^2 > 1 \tag{1}$$

(i) $\forall x, y\ x + y^2 > 1$. This is false. Take $x = 0$, $y = 1$ as a counterexample.

(ii) $\exists x, y$ such that $x + y^2 > 1$. This is true. Let $x = 1$ $y = 2$.

(iii) $\forall x$, $\exists y$ such that $x + y^2 > 1$. This is true. Now, prove this for all $x \in \mathbb{R}$.

Proof.

($x = 0$) Let $y = 2$. Then plugging into (1) we get $0 + (2)^2 = 0 + 4 = 4 > 1$.

($x > 0$) In this case, let $y = x+1$. Plugging into (1) we have $(x)+(x+1)^2$. Expanding the second term we have $x + (x^2 + 2x + 1)$. This simplifies into $x^2 + 3x + 1$. Since $x > 0$, $3x > 0$ and $x^2 > 0$. Then $x^2 + 3x + 1 > 1$.

($x < 0$) In this case, let $y = x - 1$. Then plugging into (1) we have $x + (x-1)^2$. Expanding the second term gives $x + (x^2 - 2x + 1)$, and simplifying we have $(x^2 - x + 1)$. But then since $x < 0$, $x^2 > 0$ and $-x > 0$, so $x^2 - x + 1 > 1$. □

(v) $\exists x$ such that $\forall y$, $x + y^2 > 1$. This is true as well. Let $x = 2$. Then, for any $y \in \mathbb{R}$, $y^2 \geq 0$, so $2 + y^2 \geq 2 > 1$.

(vi) $\forall y$, $\exists x$ such that $x + y^2 > 1$. This is true, since for any $y \in \mathbb{R}$, $y^2 \geq 0$. Regardless of y, let $x = 2$. Then we have $2 + y^2 \geq 2 + 0 = 2 > 1$.

(vii) $\exists y$ such that $\forall x$, $x + y^2 > 1$. This is false. For any choice of y, this fails at $x = -(y^2)$, since then we have $-(y^2) + y^2 = 0 < 1$.

For the second sentential function:

x is the father of y

(i) $\forall x, y$, *x is the father of y*. This is false, since this would require that for any two people x and y chosen at random that x be y's father. But I am not my girlfriend's father.

(ii) $\exists x, y$ such that *x is the father of y*. This is true, since it only requires that there exist one father-child pair. For instance, I am my father's daughter.

(iii) $\forall x$, $\exists y$ such that *x is the father of y*. This is false, since it would require that every person x be a father to a child y. But I am not anyone's father.

(iv) $\exists x$ such that $\forall y$, *x is the father of y*. This is false, since it would suppose that there is a single "universal father". But my father is not my girlfriend's father.

(v) $\forall y$, $\exists x$ such that *x is the father of y*. This is true. Everyone has a father, or else they would not have been born. Check back on this one in a few hundred years if we ever develop cloning...

(vi) $\exists y$ such that $\forall x$, *x is the father of y*. This is false, since it would require that there be a single "universal child". But for any choice of person y, pick any other person x such that x is not their father.

Exercise 7.

Dogs have a good sense of smell.

Exercise 8.

If x is a snake, then there exists an x such that x is poisonous.

Exercise 9.

(a) Both x and y are free.

(b) In this expression x is bound since it appears first in a quantifier, but y is free.

(c) All variables are bound in this expression, since we can evaluate the truth of it now. This is true by the way - pick $z = y + 1$.

(d) In this expression y and z are bound, since none of their appearances precede their appearance in a quantifier. However x is never quantified, so it is a free variable.

(e) X and z are bound, since z occurs only after its quantification, and x is bound by an operator $(=)$ relating it to y. However, y is a free variable.

(f) In this expression, y and z occur only after quantification, so they are bound. Then x is free since evaluating the truth of the sentence will depend on the value of x.

Exercise 10.

By replacing z with y in both places of 9(e), we obtain:

$$\textit{if } x = y^2 \textit{ and } y = 0, \textit{ then, for any number } y, \ x > -y^2$$

Then in the first part *if* $x = y^2$ *and* $y = 0$, y is free, since it has not yet been quantified or assigned. However, in the second part *for any number* y, $x > -y^2$ y is bound, since it has been quantified here.

Exercise 11.

If a variable occurs in a quantifier or is otherwise bound by some other relational operator, then all subsequent appearances of the variable in the expression will be as a bound variable. If the variable occurs in an expression preceding its quantification, or in an expression where it is not quantified at all, it will be as a free variable.

Exercise 12.

We'll label the sentential function *there is a number y such that* $x = y^2$ as (1) and *there is a number y such that* $x \cdot y = 1$ as (2).

Since x in (1) is a free variable, we cannot yet evaluate the statement without obtaining more info about x. So we'll look at (2) first. In this case, y is bound by quantification, and x is bound by an operator.

We can see that for this statement to be true, it must be that $x = \frac{1}{y}$. Now let's plug this into (1) to obtain $x = \frac{1}{x^2}$. For $x = \frac{1}{x^2}$ to be true, it must be that $x^3 = 1$, which is only true for $x = 1$. Now, $x = 1$ satisfies (2) as well, since (2) then reads as *there is a number y such that* $1 \cdot y = 1$, which is true, since $y = 1$ works.

Thus, (1) and (2) are both satisfied only by the pair $x = 1$, $y = 1$.