Chapter 2 Exercises

Exercise 1.

By specifically mathematical, I can only assume that Tarski means "not containing expressions from the domain of logic", which would mean designatory functions from the fields of arithmetic and geometry. To that end, I submit: x^2, $3\frac{1}{2} + 4$, and $\sqrt{3}$.

Exercise 2.

(a). For this sentence, the logical parts are: *"for any numbers x and y"*, *"if... and..."*, and *"then there is a number z such that..."* and the mathematical parts are the expressions $x > 0$, $y < 0$, $z < 0$, $x = y \cdot z$.

(b). For this sentence the logical parts are: *"for any points A and B"*, and *"there is"*, and the mathematical expressions are *"point C, which lies between A and B and is the same distance from A as from B"*.

Exercise 3.

First, we'll give the negations: the negation of $x < 3$ is $\neg(x < 3) = x \geq 3$, and the negation of $x > 3$ is $\neg(x > 3) = x \leq 3$. Then for the expression $(x \geq 3) \wedge (x \leq 3)$, it is clear that this is only satisfied by $x = 3$.

Exercise 4.

(a). This is the exclusive or.

(b). This is the inclusive or.

The sentence "Give further examples in which the word *"or"* is used in its first or in its second meaning" is itself an example of (I presume...) the inclusive or. So to give an example of the exclusive or I submit: "Would you rather be a teddy bear or a rock?"

Exercise 5.

(a). This sentence is true, since if today is Monday, tomorrow will certainly be Tuesday (T→T). For this sentence to be false, we would have to have (T→F), but if *tomorrow is Tuesday* is false, then today is not Monday, and we have (F→F), which is still a true implication.

(b). This sentence is not universally true, since there is no way to have $T \to T$, since if *"today is Monday"* is true, then *"tomorrow is Saturday"* is false, and we have $T \to F$, a false implication. However, note that if this sentence is said on any day of the week besides Monday, we have either the case of $F \to F$ (if the day is not Friday) or $F \to T$ (if the day is Friday), and so the sentential function is satisfied on these days only. However, when appending the universal quantifier and evaluating the implication as a sentence, it is false.

(c). This is true, since *"the 25th of December is Christmas day"* is always true, so there is no way to obtain the case of $T \to F$, which is necessary for the implication to be false. However, it is clear that this implication does not make much sense in the context of ordinary speech, since the day of the week has no bearing on the truth of the date of Christmas.

(d). This obviously tracks in common speech, in the sense that the clauses are related, and are both presumed to be false, so in common speech we have the $F \to F$ case universally, which makes a true implication. However, it's hard to deduce the truth of this statement from the standpoint of mathematical logic, since it relies on a few assumptions: **(1)** That beggars have wishes, **(2)**, that people with horses could ride them (if they so chose), and **(3)** A beggar is assumed to have no horses, other than those (formerly) made of wishes. Assuming all of these conditions are true and the antecendent is true, then beggars would have horses and thus could ride them. For the $T \to F$ case to occur (i.e. for the implication to be false), it would have to be that wishes *were* horses, but beggars could still not ride them, i.e if the antecendent is true and condition **(1)** is recognized as true, but condition **(2)** is false. Condition **(3)** has no bearing on this case.

(e). This is false, since the case of $x = 6$ is a counterexample. A whole number a divides another number x if there exists a whole number b such that $x = b \cdot a$. Then $6|x$ since $6 = 6 \cdot 1$ and $2|x$ since $6 = 2 \cdot 3$. Thus, the antecedent of the implication is true. However, 12 does *not* divide 6, since $6 = 12 \cdot b$ is only satisfied by $b = \frac{1}{2}$ which is not a whole number.

(f). This is true, but in a counterintuitive way from the standpoint of ordinary language. The antecedent is false, since 18 is not divisible by 4, as $18 = 4 \cdot n$ is only satisfied by $n = 4\frac{1}{2}$, which is not a whole number. But then, if the antecedent is universally false, we can never have the case $T \to F$ occur, so the sentence itself can never be false. In fact, 18 *is* divisible by 6, as $18 = 6 \cdot 3$, so we have the case $F \to T$ universally, which is true. This is counterintuitive to ordinary speech, since the truth of the consequent does not depend on the truth of the antecedent in any way.

Exercise 6.

(a). *If the angles of a triangle are congruent then it is an equilateral triangle.* Further paraphrased: *Tringles with congruent angles are equilateral.*

(b). *If x is divisible by 6, then x is divisible by 3.* Further paraphrased: *numbers divisible by 6 are divisible by 3.*

Exercise 7.

$x \cdot y > 4$ is a *necessary* condition for the validity of $x > 2$ *and* $y > 2$, but it is not a sufficient condition. We can see this by assuming $x > 2$ *and* $y > 2$ to be true. Then $x \cdot y > 2 \cdot y > 2 \cdot 2 > 4$ and so we have $(x > 2) \wedge (y > 2) \to x \cdot y > 4$.

However, it is not a sufficient condition, since the converse implication is not true. This can be seen from the counterexample of picking $x = 5$ and $y = 1$. Then $x \cdot y = 5 \cdot 1 = 5 > 4$, but $x > 2 \wedge y > 2$ is false.

Exercise 8.

(a). *x being divisible by 10 is a necessary and sufficient condition for x being divisible by 2 and x being divisible by 5.*

(b). *a quadrangle is a parallelogram if and only if the point of intersection of its diagonals are also the midpoint of each diagonal.*

As far as an example from arithmetic goes, I submit: *We say a number n is prime if and only if n is not divisible by any numbers besides n and 1.*

Exercise 9.

(a). ? Come back to this when I learn geometry..

(b). This is true, since $x^2 > 0$ for all $x > 0$ or $x < 0$. We'll show both directions of the implication are true.

(→). If $x \neq 0$ then $x > 0$ or $x < 0$. If $x > 0$, then $x^2 = x \cdot x > 0 \cdot 0 = 0$. If $x < 0$, then write $x = (-n)$ for $n > 0$. Then $x^2 = (-n)^2 = (-n) \cdot (-n) = (-1) \cdot (-1) \cdot n^2 = 1 \cdot n^2 > 0^2 = 0$.

(←). If $x^2 > 0$, then $x \cdot x > 0$, so $x \neq 0$, since if $x = 0$, then we would have $x \cdot x = 0 \cdot 0 = 0$, but this is not the case.

(c). This is false, since the converse direction is not true. A quadrangle with all right angles does not imply the quadrangle is a square. Any rectangle is a counterexample.

(d). This sentence is also false. The forward direction is true, since if x is divisible by 8, then $x = 8 \cdot n$ for some whole number n, and then $x = 2 \cdot 4n$ and $x = 4 \cdot 2n$, so x is divisible by 2 and 4 as well. However, 4 itself is a counterexample to the converse direction. 4 is divisible by 4, since $4 = 4 \cdot 1$, and 4 is divisible by 2, since $4 = 2 \cdot 2$, but 4 is not divisible by 8, since $4 = 8 \cdot n$ is only satisfied by $n = \frac{1}{2}$, not a whole number.

Exercise 10.

A definition of divisibility might go: *We say that x is divisible by y if and only if there exists a natural number n such that x is the product of y and n.*

To formulate a definition of parallel, this requires the knowledge of the terms *line* and *slope*. The definition follows: *We say that two lines A and B are parallel if and only if the slope of A and the slope of B are equivalent.*

Exercise 11.

There are a great many ways to translate these into ordinary language, owing to the many ways of formulating equivalences and implications that have been discussed. These are merely my takes.

(a). *From "if p is not the case, then p", it follows that p.*

(b). *"Not p or p" is as necessary and sufficient condition for "q being a necessary condition for p".*

(c). *It is not the case that "p or q" if and only if q follows from p.*

(d). *Not p or "q if, and only if, p implies q."*

Perhaps I have abused the exercise slightly by using quotes, but this merely serves to illustrate the difficulty in translating these expressions into ordinary English without either significantly re-arranging them, becoming extremely verbose to the point of a departure from ordinary language, or indicating some sort of grouping. In particular, it is difficult when there are nested groupings such as in (c) with $[\neg(p \vee q)]$, since it is hard to specify if the "not" refers to the first term of the disjunction, or is negating the entire disjunction.

Exercise 12.

(a). $\neg p \vee \neg q \rightarrow \neg(p \vee q)$

(b). $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \wedge q) \rightarrow r]$

(c). $[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$

Exercise 13.

Before constructing the truth tables, I'll make a note about what it means to interpret these functions as sentences. From the standpoint of sentential calculus, sentences and sentential functions have the same external appearance. The only difference is that a sentence is meant to be evaluable as true or false, and thus must have its variables bound by a universal quantifier. Thus, to evaluate a given sentential function as a sentence is to prefix a universal quantifier binding all previously free variables, and to then evaluate the truth of that sentence.

11. (a). This one is true.

p	$\neg p$	$\neg p \rightarrow p$	$(\neg p \rightarrow p) \rightarrow p$
T	F	T	T
F	T	F	T

(b). This one is true as well.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$	$[\neg p \vee q] \leftrightarrow (p \to q)$
T	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
F	F	T	T	T	T

(c). This one is false.

p	q	$p \vee q$	$\neg(p \vee q)$	$p \to q$	$[\neg(p \vee q)] \leftrightarrow (p \to q)$
T	T	T	F	T	F
F	T	T	F	T	F
T	F	T	F	F	T
F	F	F	T	T	T

12. (a). This one is false.

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$(p \vee q)$	$\neg(p \vee q)$	$(\neg p \vee \neg q) \to \neg(p \vee q)$
T	T	F	F	F	T	F	T
F	T	T	F	T	T	F	F
T	F	F	T	T	T	F	F
F	F	T	T	T	F	T	T

(b). This one is true.

p	q	r	$q \to r$	$p \to (q \to r)$	$(p \wedge q)$	$(p \wedge q) \to r$	$[p \to (q \to r)] \to [(p \wedge q) \to r]$
T	T	T	T	T	T	T	T
F	T	T	T	T	F	T	T
T	F	T	T	T	F	T	T
F	F	T	T	T	F	T	T
T	T	F	F	F	T	F	T
F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T
F	F	F	T	T	F	T	T

(c). This one is true.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \to r$	$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$
T	T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T	T
F	F	T	T	T	T	F	T	T
T	T	F	F	F	F	T	F	T
F	T	F	T	F	F	T	F	T
T	F	F	F	T	F	T	F	T
F	F	F	T	T	T	F	T	T

Exercise 14.

(a). Shown below.

p	$\neg p$	$\neg(\neg p)$	$\neg(\neg p) \leftrightarrow p$
T	F	T	T
F	T	F	T

(b). (i). The first of De Morgan's Laws:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q) \to \neg p \vee \neg q$
T	T	F	F	F	T	F	T
F	T	T	F	T	T	F	F
T	F	F	T	T	T	F	F
F	F	T	T	T	F	T	T

(ii). The second of De Morgan's Laws:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \to \neg p \wedge \neg q$
T	T	F	F	F	T	F	T
F	T	T	F	T	T	F	F
T	F	F	T	T	T	F	F
F	F	T	T	T	F	T	T

(c). (i). For logical multiplication with respect to addition:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$	$[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$
T	T	T	T	T	T	T	T	T
F	T	T	T	F	F	F	F	T
T	F	T	T	T	F	T	T	T
F	F	T	T	F	F	F	F	T
T	T	F	T	T	T	F	T	T
F	T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	F	T
F	F	F	F	F	F	F	F	T

(ii). For logical addition with respect to multiplication:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$	$[p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$
T	T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	F	T	F	F	F	T	F	T
T	T	F	F	T	T	T	T	T
F	T	F	F	F	T	F	F	T
T	F	F	F	T	T	T	T	T
F	F	F	F	F	F	F	F	T

Exercise 15.

(a). For the (true) original sentence *the fact that x is a positive number implies that -x is a negative number*, the conjugate sentences are:

Converse: *The fact that -x is a negative number implies that x is a positive number.* This one is true as well.

Inverse: *The fact that x is not a positive number implies =x is not a negative number.* Since the Converse is true, this one is also true.

Contrapositive: *The fact that -x is not a negative number implies x is not a positive number.* This one is true since the original is true.

(b). For the (unknown) original sentence *if a quadrangle is a rectangle, then a circle can be circumscribed about it*, the conjugate sentences are:

Converse: *If a circle can be circumscribed about a quadrangle, then the quadrangle is a rectangle* This one is ? as well.

Inverse: *If a quadrangle is not a rectangle, then a circle cannot be circumscribed about it.* Since the Converse is ?, this one is also ?.

Contrapositive: *If a circle cannot be circumscribed about a quadrangle, then the quadrangle is not a rectangle.* This one is ? since the original is ?.

Exercise 16.

This hinges on the notion of equivalence. From the truth table for $p \leftrightarrow q$ where p and q are sentences, we know where p is true, q is true also, and where p is false, q is false also. Then, since the truth of a sentence (equivalently, sentential function) in sentential calculus only depends on the truth of its sub-functions (equivalently, sub-sentences, when the variables are bound by a quantifier), then replacing any of the component sub-sentences with one of equivalent value (i.e, replacing p by q, or vice versa) will not change the truth of the sentence, since the equivalence of two sentences means they have matching truth values in all contexts. Then, taking the old and new sentence together (the original, and the one with substituted values), they are equivalent, since the truth values of these two sentences match in all cases.

In Section 10, the demonstration of the truth value of a sentence changing when $2x$ is substituted for x^2 depends on this, although in an inverted way, since the positivity of $2x$ and the positivity of x^2 are *not* equivalent. Since they are not, the example shown holds.

Exercise 17.

From an original conditional sentence $p \rightarrow q$, sentence (a) gives the converse and sentence (b) gives the inverse. Since the contrapositive is obtained by applying both converse and inverse (in either order) to a given conditional sentence, only by appling (a) and (b) together in sequence can they be used like the Law of Contraposition.

Sentence (a) is true if and only if (b) is true, although neither is necessarily true given any original $p \rightarrow q$.

Exercise 18.

(a). For sentence (a):

p	q	$p \to q$	$q \to p$	$\overbrace{(p \to q) \to (q \to p)}^{\mathbf{a}}$
T	T	T	T	T
F	T	T	F	F
T	F	F	T	T
F	F	T	T	T

(b). For sentence (b):

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$\overbrace{(p \to q) \to (\neg p \to \neg q)}^{\mathbf{b}}$
T	T	F	F	T	T	T
F	T	T	F	T	F	F
T	F	F	T	F	T	T
F	F	T	T	T	T	T

Clearly, one can see that the truth values in the final columns of each table are equivalent.

(c). For sentences (a) $\to$ (b) and (b) $\to$ (a):

$\neg p$	$\neg q$	a	b	$a \to b$	$b \to a$	$\neg q \to \neg p$	$(a \to b) \to (\neg q \to \neg p)$	$(b \to a) \to (\neg q \to \neg p)$
F	F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	F	F	F
T	T	T	T	T	T	T	T	T

Then, applying (a) then (b), or (b) then (a) yields a sentence with the same truth values as the contrapositive.

Exercise 19.

The law of hypothetical syllogism can be written as follows: $[(p \to q) \wedge (q \to r)] \to (p \to r)$. Then, labelling $p =$*yesterday was Monday* and $q =$*today is Tuesday* and $r =$*tomorrow will be Wednesday*, we can substitute p,q, and r into the law to obtain the sentence $p \to r$, in this case: *the fact that yesterday was Monday implies that tomorrow will be Wednesday.*

Exercise 20.

(1) Label $a =$ *the fact that* $\overbrace{\textit{yesterday was Monday}}^{p}$ *implies that* $\overbrace{\textit{today is Tuesday}}^{q}$ and $b =$ *the fact that* $\overbrace{\textit{today is Tuesday}}^{q}$ *implies that* $\overbrace{\textit{tomorrow is Wednesday}}^{r}$. Both a and b are accepted as true.

(2) From the rule of inference given in the exercise, we know that the conjunction of two true sentences is true, so $c = (a \wedge b) =$ must be true.

(3) Then by the rule of substitution, substitute $c = (a \wedge b) = [(p \to q) \wedge (q \to r)]$ into the law of hypothetical syllogism to obtain the true sentence $d = (a \wedge b) \to (p \to r)$.

(4) Then, the law of hypothetical syllogism is true, and so the conditional sentence d is true by the rule of substitution. Also, the antecedent c is true by step (2), and so by the rule of detachment the consequent $e = (p \to r) =$ *the fact that yesterday was Monday implies that tomorrow is Wednesday* is true as well.

□