Chapter 3 Exercises

Exercise 1.

Want to prove the theorem

$$\textit{If } x = z \textit{ and } y = z, \textit{ then } x = y$$

using only the laws

Theorem 1. *If* $x = y$*, then* $y = x$

Theorem 2. *If* $x = y$ *and* $y = z$*, then* $x = z$

Proof. By hypothesis, suppose $x = z$ and $y = z$ are true. Then, by Theorem 1, since $y = z$, we also know $z = y$. Finally, by Theorem 2, since we have $x = z$ and $z = y$, we know $x = y$ as well. □

Exercise 2.

Want to prove the theorem

$$\textit{If } x{=}y,\ y{=}z, \textit{ and } z{=}t, \textit{ then } x{=}t$$

using only 2.

Proof. By hypothesis, assume $x = y$, $y = z$, and $z = t$ are true. Then, by 2, $x = y \wedge y = z$, so $x = z$. Then, by 2 again, $x = z$ and $z = t$, so $x = t$. □

Exercise 3.

(Law III) This becomes

$$\textit{If } x \neq y, \textit{ then } y \neq x$$

which is true.

Proof. We know $x = y$ if and only if x has all the properties of y and y has all the properties of x, from Liebniz's Law. Thus, take the negation of Liebniz's law to get the sentence

$$x \neq y \leftrightarrow x \textit{ does not have all the properties that } y \textit{ does, or } y \textit{ does not have all the properties that } x \textit{ does.}$$

Similarly, to get the sentence defining $y \neq x$, interchange the variables to obtain:

$y \neq x \leftrightarrow y$ *does not have all the properties that* x *does, or* x *does not have all the properties that* y *does.*

Then, label the expression x *does not have all the properties that* y *does* as p and the expression y *does not have all the properties that* x *does* by q. Then we have the statements $p \vee q$ and $q \vee p$, which by the communtative law for logical addition are equivalent. Thus, the left hand sides of the two sentences are also equivalent: thus $x \neq y \leftrightarrow y \neq x$. □

(Law IV) This becomes

If $x \neq y$ *and* $y \neq z$*, then* $x \neq z$

which is false. We can see this from the counterexample $x = 1$, $y = 2$, $z = 1$. Then, $x \neq y$ and $y \neq z$, but $x = z$.

Exercise 4.

(a). This is true. The convention means that 0 is a number, and certainly 0 (the number) is an integer.

(b). This is false. The convention means that 0 is a number, which is a mathematical quantity and *not* a cipher having an oval shape.

(c). This is false. "0" refers to the symbol that acts as a name for the number 0. Thus "0" is not an integer.

(d). This is true. "0" refers to the symbol that acts as a name for the number 0, and it is certainly true that this symbol has an oval shape.

(e). True. These numbers are logically identical and thus arithmetically identical, since the quantities are the same. They only differ in how they are written.

(f). False. The convention around quotation marks means that "1.5" and "$\frac{3}{2}$" denote the symbols that act as names of numbers. While the numbers denoted are equal, the symbols are not.

(g). This is true. $2 + 2 = 4 \neq 5$.

(h). This is also true, the symbols "$2 + 2$" and "5" are not equal.

Exercise 5.

(a). (i). *John is a man.* This is true, since John without quotation marks refers to the man himself.

(ii). *"John" is a man.* This is false, since "John" with quotation marks refers to the name of the man.

(iii). *""John"" is a man.* This is also false, since ""John"" in double quotation marks refers to the expression used to write the name of the man.

(b). (i). *John is the name of a man.* False, John without quotation marks refers to the man himself, who as a human being cannot be a name.

(ii). *"John" is the name of a man.* This is true, since "John" in quotation marks refers to the name of the man John.

(iii). *""John"" is the name of a man.* This is false, since ""John"" refers to the expression used to write the name of the man John.

(c). (i). *John is an expression.* This is false, since John is a man, not an expression.

(ii). *"John" is an expression.* This is true, since "John" is the expression used as the name of the man John.

(iii). *""John"" is an expression.* This is also true, since ""John"" is the expression used to write the name "John".

(d). (i). *John is an expression containing quotation marks.* This is false. John is a man, not an expression.

(ii). *"John" is an expression containing quotation marks.* This one I don't quite know if the whole expression is meant to include the quotation marks or is merely an aid to the viewer. Certainly the man John would not write his name "John" on a form, nor does his driver's license read "John". Thus I would guess the answer is false. However, if we mean to regard "John" as a six-character expression, then it is true.

(iii). *""John"" is an expression containing quotation marks.* This is true. ""John"", regardless of the result of the discussion above, certainly contains quotation marks.

Exercise 6.

This consideration pops up when seeing differing formulation of conditional sentences. When formulations are labelled with logical terms such as "hypothesis", "conclusion", etc. These are referring to the arithmetical designatory functions *x is a positive number* and *2x is a positive number* as sentential functions, that is, evaluating them logically. Thus, according to our convention, it would be appropriate to add quotation marks as follows:

Thus, the hypothesis "x is a positive number" implies the conclusion "2x is a positive number"

The condition "x is a positive number" is sufficient for 2x to be a positive number.

The condiiton "2x is a positive number" is necessary for x to be a positive number.

Exercise 7.

To be proper, one might say that variables occuring in sentential calculus stand for the names of sentences or sentential functions, and thus that to evaluate universal theorems about p and q in the sentential calculus is to actually evaluate statements about the names of arbitrary sentences.

Exercise 8.

(Isoceles). If a is the base of the triangle, with b and c being the sides of equal length, then the height of the triangle is the altitude h_a. The median m_a then is the same line perpendicular to a and passing through angle A, as is bisector s_a. Thus, $h_a \cong m_a \cong s_a$, but also $h_a = m_a = s_a$, since these are all the same line. Now, while b and c are of equal length, and angles B and C are of equal degree, these two sides are mirror images of each other. Thus $h_b \cong h_c$, $m_b \cong m_c$, and $s_b \cong s_c$, but $h_b \neq h_b$, $m_b \neq m_c$, and $s_b \neq s_c$.

(Equilateral). In an equilateral traingle (considered here as a special case of an isoceles triangle), we have the same case with side a, namely that $h_a = m_a = s_a$. But since a, b, and c are all of equal length, and only differ from each other by rotation, this is true for the other sides as well: $h_b = m_b = s_b$ and $h_c = m_c = s_c$. Further, while not equal (since they differ by rotation), all altitutes, medians, and bisectors are congruent: $h_a \cong h_b \cong h_c$, $m_a \cong m_b \cong m_c$, and $s_a \cong s_b \cong h_c$.

Exercise 9.

(a). The clause *there are at most two things* can be rewritten as *There exists x and y such that both x and y satisfy the given condition, and for any z such that z also satisfies the given condition, then x=z or y=z.*

(b). This can be rewritten as the case given above in conjunction with the stipulation that $x \neq y$: *There exists x and y such that both x and y satisfy the given condition and $x \neq y$, and for any z such that z also satisfies the given condition, then x=z or y=z.*

Exercise 10.

(a). This is true. Rewrite our sentence as *there exists x such that $x + 3 = 7 - x$, and if there exists another n such that $n + 3 = 7 - n$, then $x = n$.* Then, simplify the expression: $x + 3 = 7 - x \to 2x + 3 = 7 \to 2x = 4 \to x = 4$. Thus, this equation is only satisfied by $x = 2$ and no other number.

(b). This is false. Rewrite our sentence as *there exist x and y such that both x and y satisfy the equation $x^2 + 4 = 4x$ and $x \neq y$, and for any z such that z satisfies the same equation, then $z = x$ or $z = y$.* Then, simplify $x^2 + 4 = 4x \to x^2 = (4x - 4) \to x^2 - (4x - 4) = 0 \to x^2 - 4x + 4 = 0 \to (x - 2)(x - 2) = 0$. Then, this equation is satisfied by $x = 2$ only, so this sentence is false.

(c). The truth of this sentence depends on what set of numbers we are considering. Rewrite our sentence as: *For any x, y and z, if x and y satisfy the inequality $y + 5 < 11 - 2y$, and if z satisfies the inequality, then x=z or y=z.* Simplify the inequality: $y + 5 < 11 - 2y \to 3y + 5 < 11 \to 3y < 6$. Regardless, $y = 0$ and $y = 1$ satisfy this. If $y \in \mathbb{Z}$, then any $y < 0$ also satisfies this inequality, so there are more than two such y and it is false. Otherwise, if $y \in \mathbb{N}$ satisfies this inequality, then $y \leq 2$, $3y < 3 \cdot 2 = 6 \not< 6$, so y must be 0 or 1.

(d). This is false. Rewrite our sentence as *There exist x, y, and z such that $x \neq y$, $y \neq z$, and $x \neq z$ and each of x, y and z satisfies the inequality $n^2 < 2n$.* Clearly, $x = 1$ is a solution, since $1^2 = 1 < 2 \cdot 1 = 2$. Are there any other solutions? If $n \in \mathbb{Z}$, then any $n < 0$ is not a solution, since then $2n < 0$ but $n^2 > 0$. 0 is not a solution, since $0 \not< 0$. Then, any $n \geq 2$ has $n^2 = n \cdot n \geq 2 \cdot n$, so this is not a solution either. Thus there is only one solution.

(e). This is true. Rewrite our sentence: *For any x, there exists y satisfying $x + y = 2$, and for any y and z such that y and z both have $x + y = 2$ and $x + z = 2$, then $y = z$.* First, we can see that y exists by letting $y = 2 - x$. Then $x + (2 - x) = (x - x) + 2 = 2$. Now, show that any z satisfying the same condition equals y: if $x + z = 2$, then $z = 2 - x = y$.

(f). This is false. Rewrite our sentence: *For any x, there exists y satisfying $x \cdot y = 3$, and for any y and z such that y and z both have $x \cdot y = 3$ and $x \cdot z = 3$, then $y = z$.* This is easily disproven by taking $x = 0$. Then for any choice of y, $x \cdot y = 0 \neq 3$.

Exercise 11.

This can be expressed by the following sentence: *There are exactly two numbers x such that $x^2 - 5x + 6 = 0$.*

Exercise 12.

In this sentential function, x is a free variable and y is a bound variable, since this sentential function is evaluable as a sentence once the variable x is bound by a universal quantifier, while y requires no modification in this case. Thus, numerical quantifiers *do* bind variables. We can see this by affixing *for all x* to the sentential function to get the sentence *for all numbers x, there are exactly two numbers y such that $x = y^2$.* Then, this sentence is true for all square numbers x such that $x = n \cdot n$ for some $n \in \mathbb{Z}$. Then $y = n$ and $y = -n$ are the two numbers that work. It is not true for $x = 0$ or $x = 0$. For $x = 0$, $0 = y^2$ only if $y = 0$, and since any y^2 is positive for $y \in \mathbb{Z}$, if $x < 0$ then $x \neq y^2$ for all y.