Chapter 4 Exercises

Exercise 1.

Given $K = \{x : x < \frac{3}{4}\}$,

($0 \in K$): This is true, since $0 < \frac{3}{4}$.

($1 \in K$): This is false, since $1 > \frac{3}{4}$.

($\frac{2}{3} \in K$): This is true, since $\frac{2}{3} < \frac{3}{4}$.

($\frac{3}{4} \in K$): This is false, since $\frac{3}{4} \not< \frac{3}{4}$

($\frac{4}{5} \in K$): This is false, since $\frac{4}{5} > \frac{3}{4}$

Exercise 2.

Let A be the set of all positive numbers, B be the set of all numbers less than 3, C be the set of all numbers x such that $x + 5 < 8$, and D be the set of all numbers x satisfying the relation $x < 2x$.

We want to find out which of these four sets are identical and distinct from one another. First, A is distinct from B and C since $4 \in A$ but $4 \notin B$ (as $4 > 3$) and $4 \notin B$ (since $4 + 5 = 9 > 8$). However $A = D$ since $A \subset D \land A \supset D$:

($A \subset D$): Let $x \in A$. Then $x > 0$, so $2x = x + x > x$ and $x \in D$.

($A \supset D$): Let $x \in D$. Then $2x > x$ so $x \neq 0$ (since $0 \not> 0$) and $x \not< 0$ (since then $2x < x$), so we must have $x > 0$ and $x \in A$.

Then, $A = D$, so $D \neq B$ and $D \neq C$ also. Thus, the only remaining thing to investigate is whether B and C are identical. They are:

($B \subset C$): Let $x \in B$. Then $x < 3$. If $x \leq 0$, then $x + 5 \leq 5 < 8$ and $x \in C$. Otherwise if $x = 1$, then $1 + 5 = 6 < 8$, so $x \in C$ and similarly if $x = 2$ since $2 + 5 = 7 < 8$.

($B \supset C$): Let $x \in C$. Then $x + 5 < 8$. Simplifying this expression we have $x < 8 - 5 \to x < 3$. Then $x \in B$.

Since $B \subset C$ and $B \supset C$, $B = C$.

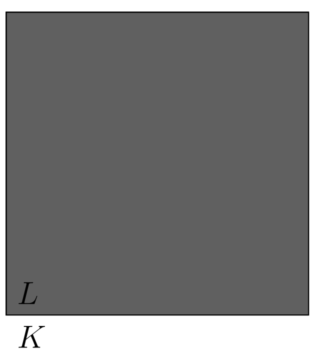
Exercise 3.

This refers to the *area* of a region. For example, from a givcen point, the set of all points not exceeding the distance of a given line segment defines a circular area surrounding the point, and if we take all the points less than the distance of a given line segment from another line segment, we produce a rectangular area.

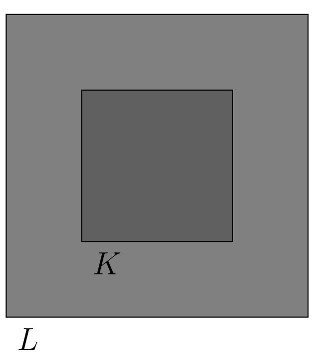
Exercise 4.

If K and L are circles with a shared center point with radii $r_K < r_L$, then we might say that the set of points L has $L \supset K$, since every point of K lies within the area of L. The circumferences of K and L, however, are disjoint, since there is no overlap between them.

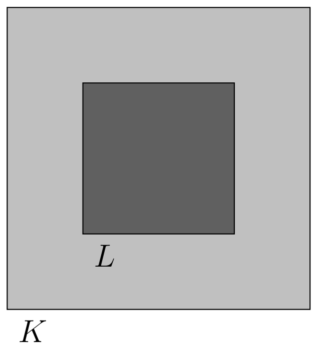
Exercise 5.



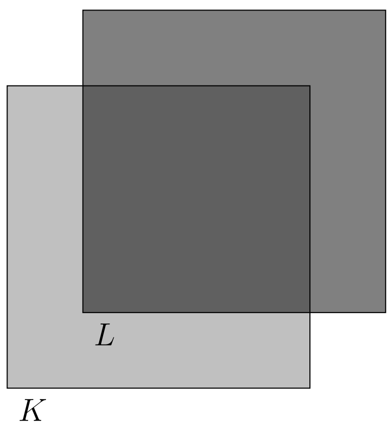
$(K = L)$:



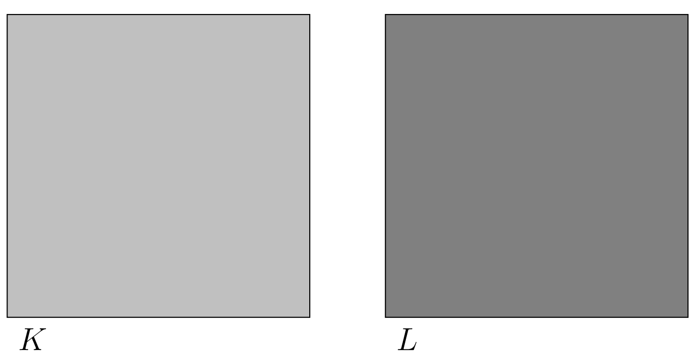
$(K \subset L)$:



$(K \supset L)$:



$(K \cap L \neq \emptyset)$:



$(K \cap L = \emptyset)$:

Exercise 6.

(a). This is true. Let $x \in [3, 5]$. Then $x \geq 3$ and $x \leq 5$. Then $6 > 5$, so $x \leq 5 < 6$ so $x \in [3, 6]$.

(b). This is false. A counterexample is $4 \in [4, 7]$ but $4 \notin [5, 10]$.

(c). This is also false, since $5 \in [-3, 5]$ ut $5 \notin [-2, 4]$.

(d). This is true. Let $x \in [-5, -2]$. Then $x \leq -2$ and $x \geq -5$. Then $-5 \geq -7$ so $x \geq 5 > 7$ and $-2 < 1$ so $x \leq -2 < 1$ and $x \in [-7, 1]$.

(e). The intervals $[2, 4]$ and $[5, 8]$ are disjoint: if $x \in [2, 4]$ then $x \geq 2$ and $x \leq 4$. But then $x \leq 4 < 5$ so $x \geq 5$ cannot be true and $x \notin [5, 8]$. Also if $x \in [5, 8]$ then $x \geq 5$, so $x \geq 5 > 4$ so $x \leq 4$ cannot be true and $x \notin [2, 4]$.

(f). $[3, 6] \supset [3\frac{1}{2}, 5\frac{1}{2}]$. If $x \in [3\frac{1}{2}, 5\frac{1}{2}]$, $x \geq 3\frac{1}{2}$ and $x \leq 5\frac{1}{2}$. Then $x \leq 5\frac{1}{2} < 6$ and $x \geq 3\frac{1}{2} > 3$ so $x \in [3, 6]$. However $[3, 6] \not\subset [3\frac{1}{2}, 5\frac{1}{2}]$ since $6 \in [3, 6]$ but $6 \notin [3\frac{1}{2}, 5\frac{1}{2}]$.

(g). $[1\frac{1}{2}, 7]$ and $[-2, 3\frac{1}{2}]$ intersect. They each have distinct elements: $-2 \in [-2, 3\frac{1}{2}]$ but $-2 \notin [1\frac{1}{2}, 7]$ and $7 \in [1\frac{1}{2}, 7]$ but $7 \notin [-2, 3\frac{1}{2}]$. However, $2 \in [1\frac{1}{2}, 7]$ and $2 \in [-2, 3\frac{1}{2}]$.

Exercise 7.

No. We can see this by letting $K = M$. Then $K \cap L = \emptyset$ and $K \cap M = \emptyset$ but $K \cap M = K = M$.

Exercise 8.

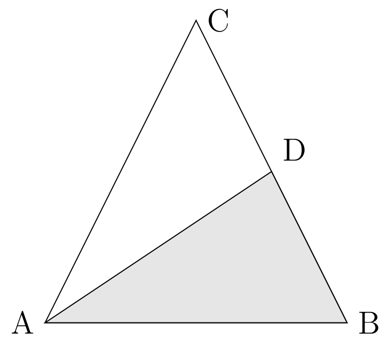
(a). The formula $(x = y) \leftrightarrow \forall K[(x \in K) \leftrightarrow (y \in K)]$ can be translated *x equals y if and only if for all classes K, x is a member of K if and only if y is a member of K*. This can be recognized as Liebniz's Law.

(b). The formula $(K = L) \leftrightarrow \forall x[(x \in K) \leftrightarrow (x \in L)]$ can be translated *K is identical to L if and only if for all individuals x, x is an element of K if and only if x is an element of L*. This is the second law given as an example, which states that K and L are identical if they are subclasses of each other.

To arrive at a definition for $K \subset L$ instead of $K = L$, we would change the bracketed equivalence on the right hand side to be an implication instead: $(K \subset L) \leftrightarrow \forall x[(x \in K) \rightarrow (x \in L)]$. Similarly, to get a definition for $K \supset L$ we take the converse of this implication: $(K \supset L) \leftrightarrow \forall x[(x \in L) \rightarrow (x \in K)]$.

Exercise 9.

The figure below represents the situation.

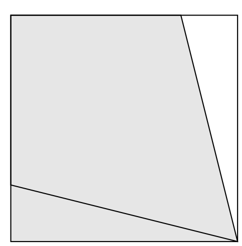


Then, we can see that $ABD \cup ACD = ACD$ (since no new points are added to ACD, as ABD completely lies within ACD) and $ABD \cap ACD = ABD$ (ABD completely lies within ACD so no points are subtracted).

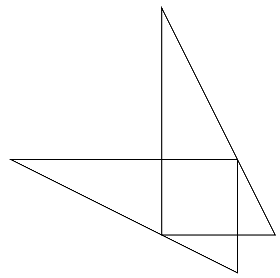
Exercise 10.

These two exercises I believe are meant to be represented by pictures to help build visual intuition of the notions of the intersection and sum of classes.

(a). The picture displated below represents an arbitrary square in the manner described. It might be a little hard to parse, but this represents two right trapezoids.



(b). The picture displated below represents an arbitrary square in the manner described.



Exercise 11.

(a). This is true. We'll demonstrate the equality by proving $[2, 3\frac{1}{2}] \cup [3, 5] \subset [2, 5]$ and $[2, 3\frac{1}{2}] \cup [3, 5] \supset [2, 5]$.

($\subset$): Let $x \in [2, 3\frac{1}{2}] \cup [3, 5]$. Then $x \in [2, 3\frac{1}{2}]$ so $x \geq 2$, and since $x \in [3, 5]$ we know $2 < 3 \leq x \leq 5$. Then $2 \leq x \leq 5$ so $x \in [2, 5]$.

($\supset$): Let $x \in [2, 5]$. Then $2 \leq x \leq 5$. Then, if $x \geq 3$, $x \in [3, 5]$ and otherwise $2 \leq x < 3 < 3\frac{1}{2}$ so $x \in [2, 3\frac{1}{2}]$.

(b). This equality is false. As a counterexample, take $-1 \in [-1, 2] \cup [0, 3]$. However, $-1 \notin [0, 2]$. In actuality, $[-1, 2] \cup [0, 3] = [-1, 3]$. Show this by showing both sides of this equation contain one another:

($\subset$): Let $x \in [-1, 2] \cup [0, 3]$. Then either $-1 \leq x \leq 2$ and so $-1 \leq x < 3$ and $x \in [-1, 3]$, or $0 \leq x \leq 3$ and so $-1 < 0 \leq x \leq 3$ and $x \in [-1, 3]$.

($\supset$): Let $x \in [-1, 3]$. Then, if $x < 0$, we have $-1 \leq x < 0$ and so $x \in [-1, 2] \subset [-1, 2] \cup [0, 3]$. Otherwise, $0 \leq x \leq 3$ and $x \in [0, 3] \subset [-1, 2] \cup [0, 3]$.

(c). This equality is also false. As a counterexample, take $-2 \in [-2, 8]$. Then $x \notin [3, 7]$ since $-2 < 3$ so $-2 \notin [-2, 8] \cap [3, 7]$. In actuality, $[-2, 8] \cap [3, 7] = [3, 7]$:

($\subset$): Let $x \in [-2, 8] \cap [3, 7]$. Then $-2 \leq x \leq 8$ and $3 \leq x \leq 7$. We notice that $-2 < 3 \leq x \leq 7 < 8$, so $x \in [3, 7]$.

($\supset$): Let $x \in [3, 7]$. Then, clearly $x \in [3, 7]$. But is $x \in [-2, 8]$ also? Yes, since $-2 < 3 \leq x \leq 7 < 8$, so $x \in [-2, 8]$ and $x \in [-2, 8] \cap [3, 7]$.

(d). This equality is also false. Take $x \in [2,3]$. Then $2 \le x \le 3$. But then, if $x \in [2, 4\frac{1}{2}] \cap [3,5]$ also, we must have $3 \le x \le 5$, so $x \le 3$ and $x \ge 3$. Thus $x = 3$. This means that any $x \neq 3 \in [2,3]$ has $x \notin [2, 4\frac{1}{2}] \cap [3,5]$. In actuality, $[2, 4\frac{1}{2}] \cap [3,5] = [3, 4\frac{1}{2}]$:

($\subset$): Let $x \in [2, 4\frac{1}{2}] \cap [3,5]$. Then $2 \le x \le 4.5$ and $3 \le x \le 5$, but noticing $2 < 3 \le x \le 4\frac{1}{2} < 5$ we have $x \in [3, 4\frac{1}{2}]$.

($\supset$): Let $x \in [3, 4\frac{1}{2}]$. Then, $4\frac{1}{2} < 5$ so $x \in [3,5]$ and $2 < 3$ so $x \in [2, 4\frac{1}{2}]$.

Exercise 12.

Let K and L be two arbitrary classes with $K \subset L$. Here, $\emptyset$ and $\mathbb{U}$ will denote the null and universal classes, respectively.

($K \cup L$): $K \cup L = L$. Take $x \in K \cup L$. Then either $x \in K \subset L$ or $x \in L$, so either way $x \in L$. Now take $x \in L$. Then $x \in K \cup L$.

($K \cap L$): $K \cap L = K$. Take $x \in K \cap L$. Then $x \in K$. Now take $x \in K$. Then $K \subset L$ so $x \in L$ also, and $x \in K \cap L$.

($K \cup \mathbb{U}$): Taking $L = \mathbb{U}$, the same relation applies (since $K \subset \mathbb{U}$ for any class K), so $K \cup \mathbb{U} = \mathbb{U}$. See proof above.

($K \cap \mathbb{U}$): Taking $L = \mathbb{U}$, the same relation applies (since $K \subset \mathbb{U}$ for any class K), so $K \cap \mathbb{U} = K$. See proof above.

($K \cup \emptyset$): Taking $K = \emptyset$, the same relation applies (since $\emptyset \subset K$ for any class K), so $\emptyset \cup L = L$. See proof above.

($K \cap \emptyset$): Taking $K = \emptyset$, the same relation applies (since $\emptyset \subset K$ for any class K), so $\emptyset \cap L = \emptyset$. See proof above.

Exercise 13.

For any classes K, L, M:

(a). $K \subset K \cup L$ and $K \supset K \cap L$:

($K \subset K \cup L$): Take $x \in K$. Then $x \in K \cup L$ since the property $x \in K \vee x \in L$ is true.

($K \supset K \cap L$): Take $x \in K \cap L$. Then $x \in K$ and $x \in L$, so $x \in K$.

(b). $\overbrace{K \cap (L \cup M) = (K \cap L) \cup (K \cap M)}^{i}$ and $\overbrace{K \cup (L \cap M) = (K \cup L) \cap (K \cup M)}^{ii}$:

(i): We'll prove each side contains the other:

($\subset$): Let $x \in K \cap (L \cup M)$. Then $x \in K$ and $x \in L \cup M$, so $x \in L$ or $x \in M$. If $x \in L$, then $x \in K \cap L$. If $x \in M$, then $x \in K \cap M$. Thus, either way $x \in (K \cap L) \cup (K \cap M)$.

($\supset$): Let $x \in (K \cap L) \cup (K \cap M)$. Then $x \in (K \cap L)$ or $x \in (K \cap M)$. If $x \in K \cap L$, then $x \in K$ and $x \in L$. If $x \in K \cap M$, then $x \in K$ and $x \in M$. In either case, $x \in K$, and all that's left is whether $x \in L$ or $x \in M$. Then, no matter what, $x \in K$ and $x \in L \cup M$, so $x \in K \cap (L \cup M)$.

(ii): Again, we'll prove each side contains the other:

($\subset$): Let $x \in K \cup (L \cap M)$. Then $x \in K$ or $x \in L \cap M$. If $x \in K$, then $K \subset (K \cup L)$ by part (a) above, so $x \in (K \cup L)$. Similarly, $K \subset (K \cup M)$, so $x \in (K \cup M)$. Then for $x \in K$, $x \in (K \cup L) \cap (K \cup M)$. If $x \in (L \cap M)$, then $x \in L$ and $x \in M$. Again by part (a), $L \subset (K \cup L)$ and $M \subset (K \cup M)$, so $x \in (K \cup L) \cap (K \cup M)$.

($\supset$): Let $x \in (K \cup L) \cap (K \cup M)$. Then $x \in (K \cup L)$, so $x \in K$ or $x \in L$. Also, $x \in (K \cup M)$, so $x \in K$ or $x \in M$. If $x \notin K$, then $x \in L$ and $x \in M$, so $x \in L \cap M$, and by part (a), $(L \cap M) \subset K \cup (L \cap M)$, so $x \in K \cup (L \cap M)$. Otherwise, $x \in K$, and again by part (a), $K \subset K \cup (L \cap M)$, so $x \in K \cup (L \cap M)$.

(c). $(K')' = K$:

($\subset$): Let $x \in (K')'$. Then $x \notin K'$. We know $K' \cup K = \mathbb{U}$. Then $x \in \mathbb{U}$, so $x \in K$ or $x \in K'$. Since $x \notin K$, it must be that $x \in K$.

($\supset$): Let $x \in K$. Then $x \notin K'$, so $x \in (K')'$.

(d). $\overbrace{(K \cup L)' = K' \cap L'}^{i}$ and $\overbrace{(K \cap L)' = K' \cup L'}^{ii}$:

(i): We'll prove each side contains the other:

($\subset$): Let $x \in (K \cup L)'$. Then $x \notin (K \cup L)$ so $x \notin K$, since by (a) $K \subset (K \cup L)$ and similarly $x \notin L$. Then $x \in K'$ and $x \in L'$. Thus $x \in K' \cap L'$.

($\supset$): Let $x \in K' \cap L'$. Then $x \notin K$ and $x \notin L$. Then $x \notin K \cup L$ so $x \in (K \cup L)'$.

(ii): Again, we'll prove each side contains the other:

($\subset$): Let $x \in (K \cap L)'$. Then $x \notin K \cap L$. Thus $x \notin K$ or $x \notin L$, so $x \in K'$ or $x \in L'$. By part (a), $K' \subset (K' \cup L')$ and $L' \subset (K' \cup L')$, so either way $x \in (K' \cup L')$.

($\supset$): Let $x \in K' \cup L'$. Then $x \in K'$ or $x \in L'$. Then if $x \notin K$, $x \notin K \cap L$ so $x \in (K \cap L)'$. But if $x \notin L$, $x \notin K \cap L$ so $x \in (K \cap L)'$ also.

Exercise 14.

The similary between the laws of sentential calculus and the laws of the calculus of classes lies in two places. First being the ways that these laws are formulated: since they take the form of sentential functions, usually of equivalences or implications, then the truth of these laws depends on the truth of these sub-functions. Thus, substituting true functions of sentential calculus or calculus of classes can result in similarly formed laws.

The second cause of the similarity comes from the properties shared by the operations utilized in each of these fields. The operations $\wedge$ and $\cap$, for example, both bear a similarity to multiplication, in that 0 (false or $\emptyset$) extinguishes 1 (true or $\mathbb{U}$), and that they are commutative, transitive, and distribute over addition. Similiarly, $\vee$ and $\cup$ bear a similarity to addition in that 0 does *not* extinguish 1, and are commutative, transitive, and distrubute over multiplication. Further, the opposition of the basic properties True vs False and $x \in K$ vs $x \notin K$ means that very similar statements can be constructed using these operations.

The law of contraposition for the calculus of classes can be forumlated in symbols as $[(x \in P) \to (x \in Q)] \to [(x \in Q') \to (x \in P')]$.

Exercise 15.

(a). The universal class:

(i). $\mathbb{U} = \underset{x}{\mathbf{C}}[\exists x]$

(ii). $x \in \mathbb{U} \leftrightarrow \exists x$

(b). The null class:

(i). $\emptyset = \underset{x}{\mathbf{C}}[x \in \mathbb{U}']$

(ii). $x \in \emptyset \leftrightarrow x \in \mathbb{U}'$

(c). The product of two classes:

(i). $K \cap L = \underset{x}{\mathbf{C}}[x \in K \wedge x \in L]$

(ii). $x \in K \cap L \leftrightarrow (x \in K \wedge x \in L)$

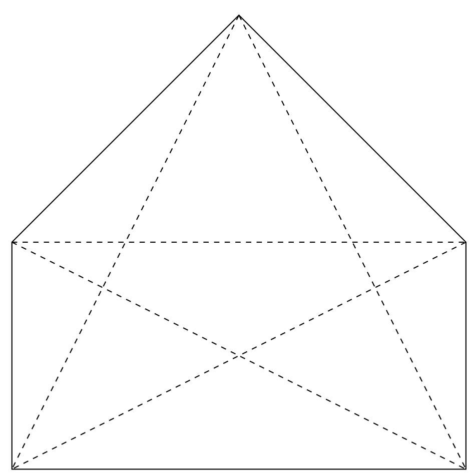
(d). The complement of a class:

(i). $K' = \underset{x}{\mathbf{C}}[x \notin K]$

(ii). $x \in K' \leftrightarrow x \notin K$

Exercise 16.

This statement is true for a pentagon. This image might demonstrate that:



Exercise 17.

(a). The expression *"the class K consists of two elements"* might be rendered:

We say the class K consists of two elements if and only if $\forall x, y, z \in K,\ (x = z \vee y = z) \wedge (x \neq y)$.

(b). The expression *"the class consists of three elements"* might be rendered:

We say the class K consists of three elements if and only if $\forall w, x, y, z \in K, [(w = x) \vee (w = y) \vee (w = z)] \wedge [(x \neq y) \wedge (y \neq z) \wedge (x \neq z)]$.

Exercise 18.

(a). This set consists of all $n \in \mathbb{N}$ such that $0 < n < 4$, i.e. the set $1, 2, 3$. Then we can say that this is equinumerous with the subset of $n \in \mathbb{N}$ such that $n < 3$, i.e. the set 0,1,2. We can see this by pairing off 0 with 1, 1 with 2, and 2 with 3. Thus, our set is equinumerous with a subset of $\mathbb{N}$, so it is finite.

(b). This set is infinite. To prove this, we'll first suppose it was finite. Then our class K would have n elements for some $n \in \mathbb{N}$: that is, $K = \{a_0, a_1, ..., a_{n-1}\}$ where each $a_i \in \mathbb{Q}$ has $0 < a_i < 4$ and $a_{i-1} < a_i$. Then construct a new element $a_n = \frac{4+a_{n-1}}{2}$. Then, $a_n \in \mathbb{Q}$ since it is a ratio of two integers, and $a_n < 4$ since $a_n - 1 < 4$, so $a_n < \frac{4+4}{2} = 4$. Then $a_n \in K$. But this means that K actually must contain at least $n + 1$ elements, which contradicts our assumption that K contains exactly n elements.

(c). This set is also infitite. I don't believe that Tarski has provided us the tools by which to prove this proposition (a definiton of the irrational numbers, notions of countability or uncountability, decimal expansions of real numbers, etc.), so I will not attempt to.