Chapter 5 Exercises

Exercise 1.

arithmetic: Divisibility: we say xRy, or x *divides* y if $\exists n \in \mathbb{N}$ such that $y = x \cdot n$

geometry: The relation of being complementary angles: $\angle a$ and $\angle b$ have $\angle a\ R \angle b$ if the sum of $\angle a$ and $\angle b$ is 90°.

physics: The relation of velocity $v = \frac{\Delta d}{\Delta t}$, where Δd is the change in displacement of an object and Δt is the change in time.

real life: The relation of being sisters: xSy if x and y are sisters.

Exercise 2.

The domain of the relation F is *"all x such that $\exists y$ such that x is the father of y"*. That is, the domain is all fathers. Not every human is a father, so the domain is not all of humanity. However, every human *does* have a father, so all of humanity does belong to the co-domain.

Exercise 3.

(a). Defining names for familial relations:

($\breve{B}$): The relation xBy applies when x is a brother to y. We know when xBy, $y\breve{B}x$, so we could say $\breve{B}$ is when y *has* a brother x. Thus $\breve{B}$ is the relation of *having a brother.*

($\breve{H}$): The relation xHy applies when x is a husband to y. We know when xHy, $y\breve{H}x$, so we might say that y has a husband x. Tarski might mean for this relation to have been called wife, but given the increased popularity of gay marriage in the last hundred years, I'll call it *having a husband* or *being a spouse.*

($H \cup W$): The relation $xH \cup Wy$ is satisfied when x is a husband or a wife to y. Thus we might say that this is the relation of *being a spouse.*

($F \cup B$): The relation $xF \cup By$ is satisfied when x is the father of y or x is the brother of y. This relation has no simple name in English.

(F/M): The relation $x(F/M)y$ is satisfied when $\exists z$ such that xFz and zBy, that is: when x is the father of z, who is the mother of y. This is the relation of x being maternal grandfather to y.

$(M/\breve{C})$: The relation $x(M/\breve{C})y$ is satisfied when $\exists z$ such that xMz and $z\breve{C}y$. Put another way, this is when x is mother of z, who in turn has a child y. This is the relation of being grandmother (either paternal or maternal).

$(B/\breve{C})$: The relation $x(B/\breve{C})y$ is satisfied when $\exists z$ such that xBz and $z\breve{C}y$. This is when x is brother of z, who has a child y. This is the relation of being uncle by blood.

$(F/(H \cup W))$: The relation $x(F/(H \cup W))y$ is satisfied when $\exists z$ such that xFz and $z(H \cup W)y$, or, when x is the father of z and z is the spouse of y. This would mean x is the father-in-law to y.

: The relation $x(B/\breve{C}) \cup [H/(S/\breve{C})]y$ is satisfied when either $x(B/\breve{C})y$ is satisfied (i.e. x is uncle by blood to y) or when $x[H/(S/\breve{C})]y$ is satisfied. Since $(S/\breve{C})$ is the relation of being blood aunt (see similarity to blood uncle), this is the relation of x being husband to some z that is the blood aunt of y. In this case, x is also y's uncle, but related by marriage. Then the relation $(B/\breve{C}) \cup [H/(S/\breve{C})]$ is that of being uncle either by blood or by marriage.

(b). Expressing familial relations in signs:

(being parent): $x(F \cup M)y$

(sibling): $x(\breve{B} \cup \breve{S})y$

(grandchild): $x(\breve{C}/\breve{C})y$

(daughter-in-law): $x[[(H \cup W)/(\breve{M})] \wedge \neg(\breve{C})]y$

(mother-in-law): $x[M/(H \cup W)]y$

(c). Explaining and verifying formulas:

$(F \subset M')$: If xFy, then x is father to y. Then if $xM'y$, x is not a mother to y. Clearly, if x is a father to y, they cannot also be y's mother. Then this formula is true.

$(\breve{B} = S)$: This is false. Suppose we have xSy where x and y are both the other's sister. Then $x\breve{B}y$ is false.

$(F \cup M = \breve{C})$: This is true. $x(F \cup M)y$ means x is a parent to y. Then $x\breve{C}y$ if and only if yCx, i.e. if y is child to x. Then $F \cup M \subset \breve{C}$. Now, $x\breve{C}y$ means x has a child y, so then $x(F \cup M)y$ is true since this means that x is a parent to y.

$(H/M = F)$: This is false. $x(H/M)y$ means $\exists z$ such that xHz, zMy. But then x could be y's stepfather. Thus it is not necessarily true that xFy.

$(B/S \subset B)$: This is true. $x(B/S)y$ means that $\exists z$ such that xBz, zSy Then, xBy also, since x, y, and z are all siblings.

$(S \subset C/\breve{C})$: This is true. If xSy, then x is a sister to y. If $xC/\breve{C}y$, then $\exists z$ such that xCz and $z\breve{C}y$. That is, x is z's child and z is y's parent. Then if x is y's sister, this is certainly true.

Exercise 4.

For arbitrary relations R and S:

(a). The formula $R/S = S/R$ does not hold for all relations. Consider the relation xRy which holds when $x = y^2$, and consider the relation xSy which holds when $x = y + 1$. Then, $x(R/S)y$ holds when $x = (y + 1)^2$, but $x(S/R)y$ holds when $x = y^2 + 1$. Thus $R/S \neq S/R$.

(b). The formula $(R\breve{/}S) = \breve{S}/\breve{R}$ does hold for all relations. We will show this by showing $(R\breve{/}S) \subset \breve{S}/\breve{R}$ and $(R\breve{/}S) \supset \breve{S}/\breve{R}$.

($\subset$): Suppose $x(R\breve{/}S)y$. Then $y(R/S)x$, so $\exists z$ such that yRz and zSx. But then $x\breve{S}z$ and $z\breve{R}y$, so $x(\breve{S}/\breve{R})y$.

($\supset$): Suppose $x(\breve{S}/\breve{R})y$. Then $\exists z$ such that $x\breve{S}z$ and $z\breve{R}y$. Then yRz and zSx, so $y(R/S)x$, and so $x(R\breve{/}S)y$.

Exercise 5.

For individuals x, y... and relations R, S...

(Universal Relation): $xUy \leftrightarrow (\exists x \wedge \exists y)$

(Null Relation): $xNy \leftrightarrow (\neg(\exists x) \vee \neg(\exists y))$

(Inclusion): $R \subset S \leftrightarrow [(xRy) \rightarrow (xSy)]$

(Equality): $R = S \leftrightarrow [(xRy) \rightarrow (xSy)] \wedge [(xSy) \rightarrow (xRy)]$

(Sum): $R \cup S \leftrightarrow (xRy) \vee (xSy)$

(Product): $R \cap S \leftrightarrow (xRy) \wedge (xSy)$

$$\text{(Negation): } xR'y \leftrightarrow \neg(xRy)$$

$$\text{(Individual Equality): } x|y \leftrightarrow [\forall K, (x \in K \leftrightarrow y \in K)]$$

$$\text{(Individual Diversity): } xDy \leftrightarrow [\exists K, (x \in K \wedge y \notin K) \vee (y \in K \wedge x \notin K)]$$

$$\text{(Composition): } x(R/S)y \leftrightarrow [\exists z, (xRz) \wedge (zSy)]$$

Exercise 6.

(a). This is reflexive. xRx since $x = x \cdot 1$. This is transitive, since if xRy, then $y = x \cdot n$ for some $n \in \mathbb{N}$ and if yRz, then $z = y \cdot m$ for some $m \in \mathbb{N}$. Then $z = (x \cdot n) \cdot m = (n \cdot m) \cdot x$, so xRz. This is not symmetrical, since $1R2$ but $2R'1$. This is also not connected, since for any two relatively prime numbers x and y, neither xRy nor yRx holds.

(b). The relation of being relatively prime is irreflexive, since x and x have a GCD of x. It is symmetric, since xRy implies yRx (the GCD does not change). Not transitive, since $2R3$ and $3R4$ but $2R'4$. Not connected either, since 2 and 4 have neither $2R4$ nor $4R2$.

(c). This is reflexive, transitive, symmetric, since congruence can be seen as being equal in some property. However it is not connected, since there exist non-congruent polygons.

(d). This is irreflexive (since nothing can be longer than itself), and asymmetric (since if x is longer than y, y cannot be longer than x), and not connected (since any x, y with equal length will have neither xRy nor yRx). However, it is transitive: if x is longer than y and y is longer than z, then x is also longer than z.

(e). This is irreflexive (since nothing can be perpendicular to itself) and intransitive (since if x is perpendicular to y and y is perpendicular to z, x and z are parallel), and not connected (since x and any parallel y have neither xRy nor yRx). However it is symmetric, since if x is perpendicular to y, y is also perpendicular to x.

(f). This is reflexive, since $x \cap x = x \neq \emptyset$. This is symmetric, since $x \cap y \neq \emptyset \rightarrow y \cap x \neq \emptyset$. This is not transitive, since $(x \cap y \neq \emptyset) \wedge (y \cap x \neq \emptyset) \not\rightarrow (x \cap z \neq \emptyset)$. This is also not connected, since any two disjoint geometric configurations x and y have neither $x \cap y \neq \emptyset$ nor $y \cap x \neq \emptyset$.

(g). This is reflexive, since any physical event x happens simultaenously to itself. This is symmetric, since if x and y happen simultaenously, y and x happen simultaenously. This is also transitive, since if x and y are simultaneous and y and z are simultaneous, then x and z are also simultaneous. However it is not connected, since any two non simultaneous events x and y have neither xRy nor yRx.

(h). This is irreflexive, since nothing can precede itself. This is also asymmetric, since if x precedes y, y cannot precede x. This is, however, transitive, since if x precedes y and y precedes z, then x precedes z as well. This is also not connected, since any two simultaneous events x and y have neither x preceding y nor y preceding x.

(i). This is reflexive, since one is related (in the same family) to oneself. This is also symmetrical, since if x is related to y, y is also related to x. This is also transitive, since if x is related to y and y is related to z, x has a relation to z as well. This is not connected, since any two unrelated people x and z have no relation in either direction.

(j). This is irreflexive, since no one can be their own father. This is also asymmetric, since if x is y's father, y cannot also be x's father. This is also intransive, since if x is y's father and y is z's father, x is not z's father but z's grandfather. This is also not connected, since any two x and y without familial relation have no relation in either direction.

Exercise 7.

(a). We want to show that $(xRx) \vee (xR'x)$ is true. If xRx is true, then $(xRx) \vee (xR'x)$ is true. If xRx is false, then $xR'x$ is true, so $(xRx) \vee (xR'x)$ is true. Thus $(xRx) \vee (xR'x)$ is always true.

(b). We want to show that $[(xRy) \to (yRx)] \vee [(xRy) \to \neg(yRx)]$ is true. If $[(xRy) \to (yRx)]$ is true, then $[(xRy) \to (yRx)] \vee [(xRy) \to \neg(yRx)]$ is true. If $[(xRy) \to (yRx)]$ is false, then the antecedent (xRy) must be true and the consequent (yRx) must be false. But then if yRx is false, $\neg(yRx)$ must be true. Then $[(xRy) \to \neg(yRx)]$ is true. Thus $[(xRy) \to (yRx)] \vee [(xRy) \to \neg(yRx)]$ is always true.

Exercise 8.

My solution to Exercise 6 has already considered the transitivity or intransitivity of its relations. Thus I will only print the considerations for Exercise 3 here.

($\breve{B}$): This is transitive: $x\breve{B}y \wedge y\breve{B}z \rightarrow x\breve{B}z$ since x has brother y and z is y's brother.

($\breve{H}$): This is intransitive with polygamy: $x\breve{H}y \wedge y\breve{H}z \not\rightarrow x\breve{H}y$ since if x has husband y, who in turn has husband z, then this does not necessarily imply $x\breve{H}z$. However, if polygamy is not permitted, then $x = z$ and this is transitive.

($H \cup W$): Intransitive, since (assuming $x \neq z$) if x is the spouse of y and y is the spouse of z, that doesn't necessarily imply that x has z as a spouse as well.

($F \cup B$): This is intransitive, since $xF \cup By \wedge yF \cup Bz$ could have xFy and yFz. But then xFz would be false.

(F/M): This is intransitive, since if x is y's maternal grandfather, and y is z's maternal grandfather, then x cannot also be z's maternal grandfather since x would be z's maternal great-great-grandfather.

($M/\breve{C}$): If x is y's grandmother and y is z's grandmother, does this imply x is z's grandmother? No, since this would mean x is z's great-grandmother. Thus this relation is intransitive.

($B/\breve{C}$): If x is y's blood uncle and y is z's blood uncle, then $\neg[x(B/\breve{C})z]$ since x is z's great-uncle. Thus intransitive.

($F/(H \cup W)$): If x father-in-law to y and y is father in law to z, then x is not father-in-law to z. Then intransitive.

: If x is uncle to y and y is uncle to z, then x is not z's uncle, since they would be either unrelated (if x is blood uncle and y is marital uncle, or vice versa) or x would be z's great uncle (if x and y are both blood uncles). Thus intransitive.

Exercise 9.

In both cases, this is basically the process of (having defined some property-class K_x for an individual x) showing that xRy means $L_x \subset L_y$ and $L_y \subset L_x$.

(parallel): Let P represent the relation of being parallel. Then, "the lines a and b are parallel" means aPb. Then, P is reflexive (since aPa for all straight lines a), symmetric (since if aPb, bPa) and transitive (since if aPb and bPc, aPc). Then, define the class $K_a = \{b, aPb\}$. Then, any straight line x is in at least one such class (xPx, so $x \in K_x$), and at most one such class as well: $\forall x, a, b$ such that $x \in K_a$ and $x \in K_b$, then aPx and bPx. Then P is symmetric, so xPb, and since P is transitive, then aPb and bPa. Thus $a \in K_b$ and $b \in K_a$, so $K_a = K_b$. It follows that any two parallel lines share the same class: if aPb, then bPa also, so $a \in K_b$ and $b \in K_a$. But then $a \in K_a$ and $b \in K_b$ also, so since each line can be a member of at most one class, it must be $K_a = K_b$. Also, any two non-parallel lines are in different classes: if $aP'b$ then $a \notin K_b$ and $bP'a$ so $b \notin K_a$. Thus $K_a \neq K_b$. Thus, calling K_a the "direction of the straight line a", we can say that aPb thus also implies the expression "the directions of the lines a and b are identical".

(congruent): Let C represent the relation of being parallel. Then "the segments ab and cd are congruent" means $abCcd$. C is reflexive (since $abCab$), symmetric (since $abCcd \to cdCab$), and transitive (since if $abCcd \wedge cdCef$, $abCef$). Then, define the class $L_xy = \{ab, xyCab\}$. Then, every segment ab is member of at least one such class, since $ab \in L_ab$, and is member of at most one also: if $ab \in L_xy$ and $ab \in L_wz$, then $xyCab$ and $wzCab$, and thus $abCwz$. Then C is transitive, so $xyPwzW$, and so $L_xy = L_wz$. From this it also follows that any two congruent segments share classes, and that if $abC'cd$, then $L_ab \neq L_cd$. Then, call L_xy the "length of the line segment xy", we can say that $abCcd$ implies the expression "the length of the segments ab and cd are identical".

Exercise 10.

The formula $x + y = y + x$ consists of 7 signs: x, $+$, y, $=$, y, $+$, and x. Then, if E denotes the relation of being equiform, $a \in \{x, +, y, =, y, +, x\}$ has aEa, if aEb then bEa, and if aEb and bEc, then aEc. Then, we can call the class $F_a = b, aEb$ the "form of the sign a". Since E is reflexive, symmetric and transitive, it follows (as seen in Exercise 9) that each sign is in exactly one form-class, that equiform signs share classes, and non-equiform signs are in different classes. Then, for $a \in \{x, +, y, =, y, +, x\}$, there are four classes: $F_x, F_y, F_+, F_=$ corresponding to the first four signs in the expression. Then, the last three symbols have $y \in F_y$, $+ \in F_+$, and $x \in F_x$. Then, as we have already seen, E is reflexive, symmetric, and transitive. Further it is not connected, since non-equiform signs x and y have neither xEy nor yEx. As for non-equiformity (E'), it is irreflexive, since $\neg xE'x$. However, it is symmetric, since if $aE'b$, then $bE'a$ as well. It is not transitive, since if $xE'y$ and $yE'z$, xEz could still be true. It is also not connected, since any two equiform a, b have neither $aE'b$ nor $bE'a$.

Exercise 11.

In Exercise 10, I referred to the class F_a for a sign a as the "form of the sign a". Since equiform varibles are members of the same F_a, it can be said that they are equal with respect to their form.

To be more precise about the situation concerning equiform variables, for an expression $x+x$, instead of saying that the same variable occurs on either side of the sign "+", we might say instead that the variables on either side of the sign have identical form, or that they are equiform.

Exercise 12.

On page 12, there are many sentences that might be amended in light of the impreciseness in speech that was discussed in Exercise 11. One example would be the statement "in both cases where the variable 'z' appears", which might be amended to "in both cases where variables equiform to "z" appear".

On page 56, a sentence reads "it is understood that, should "x" occur at several places in the formula", which might be changed to "...should variables equiform to "x occur at several places in the formula" or similar.

To more precisely formulate the expression "sentential functions with two free variables", we might instead write "sentential functions with free variables of two forms" or similar.

Exercise 13.

We say a relation R establishes an ordering if it is asymmetric, transitive, and connected. For a point O, consider $K_O = \{C, \textit{ C is a circle centered at } O\}$. For each $C \in K_O$, let r_C be the radius of C. Now consider the relation P in the set K_O, where two circles a and b have aPb if a is a part of b, i.e. if $a \subset b$. Since each a has $a = \{x,\ O < dist(O, x) < r_a\}$, $a \subset b \leftrightarrow r_a < r_b$

(a). P is asymmetric: if aPb then $bP'a$, since if aPb, then $r_a < r_b$. In fact we cannot have bPa, since then $r_a > r_b$, but this is false.

(b). P is transitive: Suppose we have a, b, and c with aPb and bPc. Then $r_a < r_b$ and $r_b < r_c$. Then $r_a < r_c$ also, so aPc.

(c). P is connected: For any two arbitrary a, b distinct within K_O, we want to show that either aPb or bPa is always true. If aPb is true, then we're done. Otherwise, if $aP'B$, then $r_a \not< r_b$. We know $r_a \neq r_b$ since a and b are distinct, so it must be that $r_a > r_b$, so bPa.

If the circles did not lie in the same plane, then P would not establish an ordering, since it would not have the property of connectedness: for a in the xy plane and b in the yz plane, for instance, neither aPb nor bPa hold.

The case is similar for the case of non-concentric circles, since two circles a, b might be disjoint, so that neither aPb nor bPa holds.

Exercise 14.

First, placing the list of words in lexiographical order, we have:

arm, army, art, ask, car, care, sale, salt, trouble

We now attempt to give a general definition of the relation Preceding, written aPb for two words a and b. To assist in the definition, we'll say that word a is n letters long and can be written $a = \{a_0, a_1, ..., a_{n-1}\}$, and b is m letters long and can be written $b = \{b_0, b_1, ..., b_{m-1}\}$.

Then, we say aPb if, starting from $i = 0$:

(-) If $a_i < b_i$, then aPb

(-) If $b_i < a_i$, then bPa

(-) If $a_i = b_i$ then consider the case of $i = i + 1$

(-) If a_i does not exist and b_i does, then aPb. If b_i does not exist and a_i does, then bPa.

Now, we'll show P establishes an ordering, i.e. that it is asymmetric, transitive, and connected.

(a). P is asymmetric: If aPb, then $\exists a_i P b_i$ and all $j < i$ have $a_j P b_j$. Thus $bP'a$, since $b_i \leq a_i$ cannot be the case.

(b). P is transitive: If $aPb \wedge bPc$, then $\exists a_i < b_i$ and all $j < i$ have $a_j{<}b_j$ and $\exists b_h < c_h$ and all $k < h$ have $a_k{\leq}b_k$.

(c). P is connected: If a and b are both words that are not equal, then select the first place where they differ and call it index i. Then $a_i \neq b_i$ so either $a_i < b_i$ or $b_i < a_i$, so either aPb or bPa.

Exercise 15.

(a). Suppose $\forall x \in K$ that xRx. Then we want to show $\neg(xR'x)$. If $xR'x$, then we would have $\neg(xRx)$, but this is not the case. Thus $\neg(xR'x)$, so R' is irreflexive.

(converse): Suppose $\neg(xR'x)$. Then xRx is true.

(b). Suppose $xRy \rightarrow yRx$. Then, replace x by y and vice versa to get $yRx \rightarrow xRy$. Then, by the law of contraposition, this gives the true expression $\neg(xRy) \rightarrow \neg(yRx)$, which in turn can be replaced by the equivalent terms $xR'y \rightarrow yR'x$. Then R' is asymmetric

(converse): Suppose R' is symmetrical. Then $xR'y \rightarrow yR'x$, and replacing y by x and x by y we get $yR'x \rightarrow xR'y$. Then the law of contraposition gives the true sentence $xRy \rightarrow yRx$.

(c). Suppose $xRy \rightarrow \neg(yRx)$. Replace y with x in is sentence to get $xRx \rightarrow \neg(xRx)$, i.e.$xRx \rightarrow (xR'x)$ Then, $xRx \wedge xR'x$ is false and $xRx \vee xR'x$ is true, the only way for $xRx \rightarrow \neg(xRx)$ to be true is if xRx is false and $xR'x$ is true. Then R' is reflexive. Now, we to show R' is connected, which is to say that $xR'y \vee yR'x$ is true. consider any $x, y \in K$. Then either xRy or $xR'y$. If xRy, $xRy \rightarrow yR'x$ since R is asymmetrical, so $yR'x$ is true, so $xR'y \vee yR'x$ is true. Otherwise we have $xR'y$ and $xR'y \vee yR'x$ is true.

(converse): Suppose R' reflexive, connected. Then $xR'x$ and $xR'y \vee yR'x \forall x, y \in K$. Then suppose xRy. Then $xR'y$ is false, so $yR'x$ must be true. But then $xRy \rightarrow yR'x$ is true, so R is asymmetrical.

(d). Suppose for all $x, y \in K$ that $xRy \wedge yRx \rightarrow xRz$ and $xRy \vee yRx$ are true.

(converse): Suppose R' is transitive: that is $xR'y \wedge yR'z \rightarrow xR'z$.

Exercise 16.

(reflexivity): Suppose xRx. We want to show $x\breve{R}x$. This follows immediately from the property that $xRy \to y\breve{R}x$ by replacing x by y.

(irreflexivity): Suppose $\neg(xRx)$. Then $xR'x$ We want to show $x\breve{R}'x$. This follows immediately from the property that $xR'y \to y\breve{R}'x$ by replacing x by y.

(symmetry): Suppose $xRy \to yRx$. Then, if xRy we also have $y\breve{R}x$, and since we have yRx, we also have $x\breve{R}y$. Then $x\breve{R}y \to y\breve{R}x$ is a true sentence.

(asymmetry) Suppose $xRy \to \neg yRx$. Then if xRy, $y\breve{R}x$. By the law of contraposition we have $yRx \to \neg(xRy)$. Then yRx is equivalent to $x\breve{R}y$ and $\neg(xRy)$ is equivalent to $\neg(y\breve{R}x)$ so we have $x\breve{R}y \to \neg(y\breve{R}x)$.

(transitivity): Suppose $xRy \wedge yRz \to xRz$. Replace x by z and z by x to get $zRy \wedge yRx \to zRx$. Then, $zRy \wedge yRx \leftrightarrow yRx \wedge zRy$. Also, $yRx \leftrightarrow x\breve{R}y$, $zRy \leftrightarrow y\breve{R}z$, and $zRx \leftrightarrow x\breve{R}z$. Then we have $x\breve{R}y \wedge y\breve{R}z \to xRz$.

(intransitivity): Suppose $xRy \wedge yRz \to \neg(xRz)$. Replace x by z and z by x to get $zRy \wedge yRx \to \neg(zRx)$. Then, $zRy \wedge yRx \leftrightarrow yRx \wedge zRy$. Also, $yRx \leftrightarrow x\breve{R}y$, $zRy \leftrightarrow y\breve{R}z$, and $\neg(zRx) \leftrightarrow \neg(x\breve{R}z)$. Then we have $x\breve{R}y \wedge y\breve{R}z \to \neg(xRz)$.

(connectedness): Suppose $xRy \vee yRx$. If xRy then $y\breve{R}x$ and $x\breve{R}y \vee y\breve{R}x$ is true. Else we have yRx, so $x\breve{R}y$ and $x\breve{R}y \vee y\breve{R}x$ is true.

(unconnectedness): Suppose $\exists x, y \in K$ such that $xR'y \wedge yR'x$. Then $xR'y \to \neg(y\breve{R}x)$ and $yR'x \to \neg(x\breve{R}y)$ so $\neg(x\breve{R}y) \wedge \neg(y\breve{R}x)$.

Exercise 17.

(-) $R/R \subset R$ expresses transitivity, since xR/Ry means $\exists z$ such that xRz and zRy. Then, we say $R \subset S$ if $xRy \to xSy$. Then, $R/R \subset R$ would express the implication $xRz \wedge zRy \to xRy$, which we know as transitivity.

(-) $D \subset R \cup \breve{R}$ expresses connectedness since xDy means x and y are distinct individuals. Then $x(R \cup \breve{R})y$ means $xRy \vee x\breve{R}y$, i.e. $xRy \vee yRx$. Then $D \subset R \cup \breve{R}$ would express the implication $xDy \to xRy \vee yRx$, which is connectedness.

(-) $R/\breve{R} \subset I$ is to say $xR/\breve{R}y \to xIy$. Then, $xR/\breve{R}y$ means $\exists z$ such that xRz and $z\breve{R}y$. But then $z\breve{R}y \leftrightarrow yRz$, and we have $xRz \wedge yRz$. Then $R/\breve{R} \subset I$ expresses the implication $xRz \wedge yRz \to xIy$ which we know as Theorem V from Section 17.

(symmetry): This can be expressed $R \subset \breve{R}$: since $x\breve{R}y$ is equivalent to yRx, we have the implication $xRy \to yRx$.

(asymmetry): This can be expressed $R \subset \neg\breve{R}$: $\neg(x\breve{R}y)$ is equivalent to $\neg(yRx)$, so we have the implication $xRy \to \neg(yRx)$.

(intransitivity): This can be expressed $R/R \subset R'$: xR/Ry means $\exists z$ such that xRz and zRy, so we have the implication $xRz \wedge zRy \to xR'y$.

Exercise 18.

(a). To better observe the relation expressed by $2x+3y = 12$, we'll rearrange the formula to get $x = \frac{12-3y}{2}$. Then this is a function if for any y, there is at most one x that can have xRy. In other words if $x = R(y)$ and $b = R(y)$ then $x = b$, or if $\frac{12-3y}{2} = \frac{12-3y}{2}$ always. Simplifying this equality yields $2 \cdot \frac{12-3y}{2} = 2 \cdot \frac{12-3y}{2} \leftrightarrow 12 - 3y = 12 - 3y \leftrightarrow 12 - 3y - 12 = 3y \leftrightarrow 3y = 3y \leftrightarrow y = y$. Then this is a function.

(b). This is a function if $x = \sqrt{y^2}$ and $z = \sqrt{y^2}$ implies $x = z$ always. This is not the case, however, since $z = -x$ also has zRy, since $z = -x = \sqrt{y^2}$ has $(-x)^2 = (x)^2 = y^2$, so R is not a function.

(c). Suppose $x > y - 5$ and $z > y - 5$. Is it the case that $x = z$? No, since for any x such that $x > y - 5$, let $z = x + 1$. Then $z > y - 5$ also.

(d). This can be rearranged $x = y^2 - y$. Then if $x = y^2 - y$ and $z = y^2 - y$, does $x = z$? This would mean for a given y that $y^2 - y = y^2 - y$, which is true, so this is a function.

(e). If x is the mother of y and z is the mother of y, does this imply $x = z$? Speaking of the birth relation, yes, since any human y can only have one birth mother. Then this is a function.

(f). If x is the daughter of y and z is the daughter of y does this imply $x = z$? No, since it could be the case that y has two daughters x and z. Then this is not a function.

Now, for the relations in Exercise 3:

($\breve{B}$): If x has brother y and z has brother y does this imply $x = z$? No, since y could have two siblings. So not a function.

($\breve{H}$): Does $x\breve{H}y$ and $z\breve{H}y$ imply $x = z$? Tarksi probably meant the answer to be yes, since in the paradigm of monogamy y would only be husband to one person. Thus, to Tarksi, this was probably a function. In 2025, however, the situation is more complicated...

($H \cup W$): If $x(H \cup W)y$ and $z(H \cup W)y$ does this imply $x = z$? Again, Tarksi probably meant the answer to be yes, since a married y would have one spouse. So this is a function. In 2025, however, the situation is more complicated...

($F \cup B$): Not a function, since $x(F \cup B)y$ and $z(F \cup B)y$ does not imply $z = x$. It could be that x is father to y and z is brother to y. But then $x \neq z$.

(F/M): If x is maternal grandfather to y and z is maternal grandfather to y, this does imply $x = z$, since any person y can only have one maternal grandfather.

($M/\breve{C}$): Not a function, since if x is (either maternal or paternal) grandmother to y and z is (either maternal or paternal) grandmother to y, then it could be that (for instance) x is paternal grandmother to y and z is maternal grandmother to y. Then $x \neq z$.

($B/\breve{C}$): Not a function, since $xB/\breve{C}y$ and $zB/\breve{C}y$ could have x paternal uncle to y and z maternal uncle to y. Then $x \neq z$.

($F/(H \cup W)$): This was probably a function to Tarski, since if $x[F/(H \cup W)]y$ and $z[F/(H \cup W)]y$ then this means x is the father of y's spouse and z is the father of y's spouse. It was probably assumed that y would only have one spouse, so then $x = z$.

: This is not a function, since we could have $x(B/\breve{C}y)$ and $z(H/\breve{S}/\breve{C})y$. Then $x \neq z$ Since x would be a blood uncle and z would be an uncle related by marriage.

Exercise 19.

We usually assume the set of all argument values to be the set of real numbers, but it could hypothetically be any set of numbers. The set of all function values is $K = \{x | x \in \mathbb{R}, x \geq 1\}$ since for all $y \in \mathbb{R}$, $y^2 \geq 0$. Then $y^2 + 1 \geq 1$.

Exercise 20.

The functions in Exercise 18 were items (a), (d), and (e). In general, to prove a function is biunique, we'll verify that it's inverse formula is a function: replace x by y and y by x and then solve for x.

(a): First find and verify the inverse formula: Suppose xRy if $x = \frac{12-3y}{2}$. Then replace y by x and vice versa: $y = 2x + 1$. Then solve for x to get $x = \frac{12-2y}{3}$. Now verify this: we should have $xRy \leftrightarrow y\breve{R}x$, so $x = \frac{12-3y}{2} \leftrightarrow y = \frac{12-2x}{3}$. Then $y = \frac{12-2\frac{12-3y}{2}}{3} \leftrightarrow y = \frac{12-(12-3y)}{3} \leftrightarrow y = \frac{12-(12-3y)}{3} \leftrightarrow y = \frac{3y}{3} \leftrightarrow y = y$. Then this function is biunique as any given function value x permits only one argument value y.

(d):

(e): If xMy is the relation x *is the mother of* y, then $x\breve{M}y$ would mean yMx, or y *is the mother of* x. Then $x\breve{M}y$ could be phrased x *has mother* y. Suppose $x\breve{M}y$ and $z\breve{M}y$. If M is biunique, then this should imply $x = z$. However, this is not the case, since it could be that x and z are siblings with mother y.

Exercise 21.

Let xRy if $x = 3y + 1$. We want to show this is a biunique function. Replacing x by y and y by x and solving for x we get $x = \frac{y-1}{3}$. Then if xRy, we should have $y\breve{R}x$, so suppose that $y = \frac{x-1}{3}$. Then $y = \frac{3y+1-1}{3} \leftrightarrow y = \frac{3y}{3} \leftrightarrow y = y$. Then this is a biunique function.

Now, suppose $y \in [0, 1]$. What can x be? If $y_i, y_j \in [0, 1]$ with $y_i < y_j$, then $R(y_i) < R(y_j)$ since $3(y_i)+1 < 3(y_j)+1$. Then if $y = 0$, $x = 3(0)+1 = 1$. If $y = 1$, $x = 3(1) + 1 = 4$. Thus $x = R(y)$ has $x \in [1, 4]$. Then, pick any $x \in [1, 4]$, and we'll show that a unique $y \in [0, 1]$ exists such that $x = R(y)$. If $x \in [1, 4]$, let $y = \frac{x-1}{3}$. This always exists in [0,1] since $x \geq 1$ so $y \geq \frac{1-1}{3} = 0$ and $x \leq 4$ so $y \leq \frac{4-1}{3} = 1$. Then this y has $x = R(y)$: $3(\frac{x-1}{3}) + 1 = (x - 1) + 1 = x$.

Exercise 22.

Let xRy if $x = 2^y$. Then let $y \in \mathbb{R}$. First, show $x\breve{R}y$ is a function: replacing x by y and y by x and solving for x we get $x = \log_2 y$. Then, if xRy we should have $y\breve{R}x$: Consider $y = \log_2 x$. Then, we have $y = \log_2 2^y \leftrightarrow y = y$, so this is a biunique function.

Now, for $y \in R$, if $y = 0$, $x = 2^0 = 1$. If $y < 0$, $y = -m$ for some $m > 0$ and we have $x = 2^{-m} = \frac{1}{2^m} \in [0, 1]$, If $y > 0$, $x = 2^y > 1$. Thus x is always positive. Now, for any $x > 0$, we'll show $\exists y$ with $x = R(y)$: let $y = \log_2 x$. Then $x = 2^{\log_2 x} = x$. Thus, any real number y can be mapped to a positive number x (or vice versa) in a one-to-one manner.

Exercise 23.

In the class $\mathbb{N}$, Consider the relation xRy that holds if $x = 2y + 1$ is satisfied. This is a function: if $x = 2y + 1$ and $z = 2y + 1$ we should have that $x = z$. This is the case since $2y+1 = 2y+1 \leftrightarrow 2y = 2y \leftrightarrow y = y$ always. This is also biunique: replacing x by y and y by x and solving for x we get $x = \frac{y-1}{2}$. Then if xRy, we should have $y\breve{x}$. Consider $y = \frac{(2y+1)-1}{2} \leftrightarrow y = \frac{2y}{2} \leftrightarrow y = y$. Then this is biunique.

Then for $y \in \mathbb{N}$, $x = 2y + 1$ is always odd: if x was even, then $x = 2m$ for some $m \in \mathbb{M}$, but then $2m = 2y + 1$ would give $m = y + \frac{1}{2}$, so $m \notin \mathbb{N}$ - a contradiction. Thus for any x odd, $\exists y$ with $x = R(y)$: let $y = \frac{x-1}{2}$. Then $x = 2(\frac{x-1}{2}) + 1 \leftrightarrow x = x - 1 + 1 \leftrightarrow x = x$. Then this is a one-to-one mapping between the set of all natural numbers and the set of all odd numbers.

Exercise 24.

(geometry): The Pythagorean Theorem is an example. For the class of "sides of a triangle T", we say $R(a, b, c)$ if $a^2 + b^2 = c^2$.

(arithmetic): The relation of being a multiple: for the set $\mathbb{Z}$ of integers, we say x is a multiple of y and z if $x = n \cdot y + m \cdot z$ for some $m, n \in \mathbb{Z}$.

Exercise 25.

(a). If $x + y + z = 0$ and $w + y + z = 0$, then $w = y + z = x$, so this is a function.

(b). This is not a function. Let $y = 0$, $z = -1$. Then $x > -2$ and $w > -2$ does not imply $x = w$, since $x = 1$ has $0 > -2$ but $w = 2$ also has $0 > -2$.

(c). This is not a function. Suppose $x = R(y, z)$ and $w = R(y, z)$. Then this should imply $x = w$. But $x = \sqrt{y^2 + z^2}$ and $w = -\sqrt{y^2 + z^2}$ both have $x^2 = y^2 + z^2$ and $w^2 = y^2 + z^2$ even though $x \neq w$.

(d). This is a function. Suppose $x = R(y, z)$ and $w = R(y, z)$, then $x = y^2 + z^2 - 2$ and $w = y^2 + z^2 - 2$. Label $n = y^2 + z^2$. Then $x = n - 2 = w$.

Exercise 26.

(two-terms): The relation of something's weight: $W = m * 9.8\frac{m}{s^2}$ where W is weight of an object and m is its mass.

(three-terms): The relation defined in Newton's Second Law: $F = m \cdot a$ where F is force, m is the mass of the object, and a is its acceleration.

(four-terms): The relation defined as the Lorenz force: $F = q(v \times B + E)$ where F is the force, q is a given charge, v is the velocity of the charge, E is the direction of the electric field, and B is the vector signifying the direction of magnetic induction.