Chapter 6 Exercises

Exercise 1.

(a). **Theorem I:** $(K \cup K) \subset K$

Proof. In Axiom III, replace L and M by K to get

$$[(K \cup K) \subset K] \leftrightarrow (K \subset K) \wedge (K \subset K) \tag{1}$$

Then, ,

$$(K \subset K) \wedge (K \subset K) \leftrightarrow (K \subset K) \tag{2}$$

Finally, by Axiom I, $K \subset K$, so we have

$$(K \cup K) \subset K \tag{3}$$

□

(b). **Theorem II:** $K \subset (K \cap K)$

Proof. In Axiom IV, replace L and M with K to get

$$[K \subset (K \cap K)] \leftrightarrow (K \subset K) \wedge (K \subset K) \tag{1}$$

Then,

$$(K \subset K) \wedge (K \subset K) \leftrightarrow (K \subset K) \tag{2}$$

Finally, by Axiom I, $K \subset K$, so we have

$$K \subset (K \cap K) \tag{3}$$

□

(c). **Theorem III:** $[K \subset (K \cup L)] \wedge [L \subset (K \cup L)]$

Proof. From Axiom III, replace M by $K \cup L$ to get

$$[(K \cup L) \subset (K \cup L)] \leftrightarrow [K \subset (K \cup L) \wedge L \subset (K \cup L)] \tag{1}$$

Then by Axiom I, $K \cup L \subset K \cup L$ is true, so we have

$$[K \subset (K \cup L)] \wedge [L \subset (K \cup L)] \tag{2}$$

□

(d). **Theorem IV:** $[(K \cap L) \subset K] \wedge [(K \cap L) \subset L]$

Proof. In Axiom IV, replace M by $K \cap L$ to get

$$[(K \cap L) \subset (K \cap L)] \leftrightarrow [(K \cap L \subset K) \wedge (K \cap L \subset L)] \tag{1}$$

Then by Axiom I, $K \cap L \subset K \cap L$, so we have

$$[(K \cap L) \subset K] \wedge [(K \cap L) \subset L] \tag{2}$$

□

(e). **Theorem V:** $(K \cup L) \subset (L \cup K)$

Proof. In Axiom III, replace M by $L \cup K$ to get

$$[(K \cup L) \subset (L \cup K)] \leftrightarrow [[K \subset (L \cup K)] \wedge [L \subset (L \cup K)]] \tag{1}$$

Then, in Theorem III, replace K by L and L by K to get

$$[L \subset (L \cup K)] \wedge [K \subset (L \cup K)] \tag{2}$$

By the Commutative Law for logical multiplication, we have

$$[L \subset (L \cup K)] \wedge [K \subset (L \cup K)] \leftrightarrow [[K \subset (L \cup K)] \wedge [L \subset (L \cup K)]] \tag{3}$$

So we have

$$(K \cup L) \subset (L \cup K) \tag{4}$$

□

(f). **Theorem VI:** $(K \cap L) \subset (L \cap K)$

Proof. In Axiom IV, replace L by K and K by L to get

$$[M \subset (L \cap K)] \leftrightarrow [M \subset L \wedge M \subset K] \tag{1}$$

Then, in (1), replace M by $K \cap L$ to get

$$[(K \cap L) \subset (L \cap K)] \leftrightarrow [(K \cap L) \subset L \wedge (K \cap L) \subset K] \tag{2}$$

Finally, by Theorem IV, the right hand side of (2) is true, so we have

$$(K \cap L) \subset (L \cap K) \tag{3}$$

□

(g). **Theorem VII:** If $L \subset M$, then $K \cup L \subset K \cup M$

Proof. Suppose $L \subset M$. Then, in Theorem III, replace L by M to get

$$K \subset (K \cup M) \wedge M \subset (K \cup M) \tag{1}$$

Then $L \subset M$ and $M \subset (K \cup M)$, so by Axiom II,

$$L \subset (K \cup M) \tag{2}$$

Then in Axiom III, replace M by $K \cup M$ to get

$$[K \cup L \subset K \cup M] \leftrightarrow [(K \subset (K \cup M)) \wedge (L \subset (K \cup M))] \tag{3}$$

Then, by (1), (3), the right hand side is true, so we have $K \cup L \subset K \cup M$, and thus

$$\text{If } L \subset M, \text{ then } K \cup L \subset K \cup M \tag{4}$$

□

(h). **Theorem VIII:** If $L \subset M$ then $K \cap L \subset K \cap M$

Proof. Suppose $L \subset M$. Then, from Theorem IV, we have

$$K \cap L \subset K \wedge K \cap L \subset L \tag{1}$$

and thus

$$K \cap L \subset L \tag{2}$$

In Axiom II, replace K by $K \cap L$ to get

$$\text{If } ((K \cap L) \subset L) \wedge (L \subset M), \text{ then } K \cap L \subset M \tag{3}$$

Then by (1) and (3), we have

$$[(K \cap L) \subset K] \wedge [(K \cap L) \subset M] \tag{4}$$

In Axiom IV, replace M by $K \cap M$ to get

$$[K \cap M \subset K \cap L] \leftrightarrow [(K \cap M \subset K) \wedge (K \cap M \subset L)] \tag{5}$$

Then in (5), replace M by L and L by M to get

$$[K \cap L \subset K \cap M] \leftrightarrow [(K \cap L \subset K) \wedge (K \cap L \subset M)] \tag{6}$$

Then, the right hand side of (6) is true by (4), so we have

$$K \cap L \subset K \cap M \tag{7}$$

□

(i). **Theorem IX:** $[K \cap L \subset [K \cap (L \cup M)]] \wedge [K \cap M \subset [K \cap (L \cup M)]]$

Proof. In Theormen III, replace K by L, and L by M to get

$$L \subset (L \cup M) \wedge M \subset (L \cup M) \tag{1}$$

Then in Theorem VIII replace M by $L \cup M$ to get

$$[L \subset (L \cup M)] \rightarrow [(K \cap L) \subset (K \cap (L \cup M))] \tag{2}$$

Then, the left hand side of (2) is true by (1), so the right hand side is true as well. Then in Theorem VIII replace M by L and L by M to get

$$M \subset L \rightarrow (K \cap M) \subset (K \cap L) \tag{3}$$

Then in (3), replace L by $L \cup M$ to get

$$M \subset (L \cup M) \rightarrow (K \cap M) \subset (K \cap (L \cup M)) \tag{4}$$

Then the left hand side is true by (1), so the right hand side is true as well. Finally, from (2) and (4) we have

$$[K \cap L \subset [K \cap (L \cup M)]] \wedge [K \cap M \subset [K \cap (L \cup M)]] \tag{5}$$

□

(j). **Theorem X:** $(K \cap L) \cup (K \cap M) \subset [K \cap (L \cup M)]$

Proof. From Axiom III, replace K by $K \cap L$, L by $K \cap M$, and M by $K \cap (L \cup M)$ to get

$$\begin{gathered}[(K \cap L) \cup (K \cap M) \subset (K \cap (L \cup M))] \\ \leftrightarrow \\ [(K \cap L) \subset (K \cap (L \cup M)) \wedge (K \cap M) \subset (K \cap (L \cup M))]\end{gathered} \tag{1}$$

Then by Theorem IX, the right hand side is true, so the left hand side is true as well. □

Exercise 2.

(a). **Theorem XI:** $K = K$

Proof. In Definition I, substitute K for L to get

$$K = K \leftrightarrow K \subset K \wedge K \subset K \tag{1}$$

Then by Axiom I, $K \subset K$ so the right hand side of (1) is true, and so the left hand side is true as well. $\square$

(b). **Theorem XII:** If $K = L$, then $L = K$

Proof. Suppose $K = L$. Then by Definition I we have

$$K \subset L \wedge L \subset K \tag{1}$$

. Then, in Definition I, replace K by L and L by K to get

$$L = K \leftrightarrow L \subset K \wedge K \subset L \tag{2}$$

Then, the commutative law gives

$$K \subset L \wedge L \subset K \leftrightarrow L \subset K \wedge K \subset L \tag{3}$$

And so we have

$$K = L \leftrightarrow [K \subset L \wedge L \subset K] \leftrightarrow [L \subset K \wedge K \subset L] \leftrightarrow L = K \tag{4}$$

and so $L = K$. $\square$

(c). **Theorem XIII:** If $K = L$ and $L = M$, then $K = M$

Proof. Suppose $K = L$ and $L = M$. Then by Definition I, we have

$$(K \subset L \wedge L \subset K) \wedge (L \subset K \wedge K \subset L) \tag{1}$$

Then $K \subset L \wedge L \subset M$ and $M \subset L \wedge L \subset K$, so by applying Axiom II twice, we have

$$K \subset M \wedge M \subset K \tag{2}$$

Then, the right hand side of Definition I is true by (2), so we have $K = M$ $\square$

(d). **Theorem XIV:** $K \cup K = K$

Proof. In Definition I, replace K by $K \cup K$ and L by K to get

$$K \cup K = K \leftrightarrow K \cup K \subset K \wedge K \subset K \cup K \tag{1}$$

By Theorem I, $K \cup K \subset K$. Then by Theorem III, replace L by K to get

$$K \subset (K \cup K) \wedge K \subset (K \cup K) \tag{2}$$

which in turn gives $K \subset K \cup K$. Then this expression alongside Theorem I gives

$$K \cup K \subset K \wedge K \subset K \cup K \tag{3}$$

and the right hand side of Definition I is satisfied. Thus $K \cup K = K$ □

(e). **Theorem XV:** $K \cap K = K$

Proof. In Definition I, replace K by $K \cap K$ and L by K to get

$$K \cap K = K \leftrightarrow K \cap K \subset K \wedge K \subset K \cap K \tag{1}$$

Then by Theorem IV, replace L by K to get $K \cap K \subset K \wedge K \cap K \subset K$ which in turn gives simply

$$K \cap K \subset K \tag{2}$$

Then In Axiom IV, replace M and L by K to get

$$K \subset K \cap K \leftrightarrow K \subset K \wedge K \subset K \tag{3}$$

Then, by Axiom I, the right hand side of (3) is satisfied, so we have $K \subset K \cap K$. From this, alongside (2), we have

$$K \cap K \subset K \wedge K \subset K \cap K \tag{4}$$

and so by Definition I we have $K \cap K = K$. □

(f). **Theorem XVI:** $(K \cup L) = (L \cup K)$

Proof. By Theorem V, we have $K \cup L \subset L \cup K$. Additionally, replace K by L and L by K in Theorem V to get

$$L \cup K \subset K \cup L \tag{1}$$

Then from these two substitutions we have

$$(K \cup L \subset L \cup K) \wedge (L \cup K \subset K \cup L) \tag{2}$$

Then, in Definition I, replace K by $K \cup L$ and L by $L \cup K$ to get

$$[(K \cup L) = (L \cup K)] \leftrightarrow [(K \cup L \subset L \cup K) \wedge (L \cup K \subset K \cup L)] \tag{3}$$

The right hand side is true by (2), so we have $(K \cup L) = (L \cup K)$ as desired. $\square$

(g). **Theorem XVII:** $(K \cap L) = (L \cap K)$

Proof. By Theorem VI, we have $K \cap L \subset L \cap K$. Additionally, replace K by L and L by K in Theorem VI to get

$$L \cap K \subset K \cap L \tag{1}$$

Then from these two substitutions we have

$$(K \cap L \subset L \cap K) \wedge (L \cap K \subset K \cap L) \tag{2}$$

Then, in Definition I, replace K by $K \cap L$ and L by $L \cap K$ to get

$$[(K \cap L) = (L \cap K)] \leftrightarrow [(K \cap L \subset L \cap K) \wedge (L \cap K \subset K \cap L)] \tag{3}$$

The right hand side is true by (2), so we have $(K \cap L) = (L \cap K)$ as desired. $\square$

(h). **Theorem XVIII:** $K \cap (L \cup M) = (K \cap L) \cup (K \cap M)$

Proof. By Axiom V, we have

$$[K \cap (L \cup M)] \subset [(K \cap L) \cup (K \cap M)] \tag{1}$$

Then, by Theorem X, we have

$$[(K \cap L) \cup (K \cap M)] \subset [K \cap (L \cup M)] \tag{2}$$

In Definition I, replace K by $[K \cap (L \cup M)]$ and $Lby[(K \cap L) \cup (K \cap M)]$ to get

$$[K \cap (L \cup M) = (K \cap L) \cup (K \cap M)]$$

$$\leftrightarrow \tag{3}$$

$$[[K \cap (L \cup M)] \subset [(K \cap L) \cup (K \cap M)] \wedge [(K \cap L) \cup (K \cap M)] \subset [K \cap (L \cup M)]]$$

Then the right hand side is true by (1) and (2), so we have $K \cap (L \cup M) = (K \cap L) \cup (K \cap M)$ as desired. □

(i). **Theorem XIX:** $K \cup K' = \mathbb{U}$

Proof. By Axiom VIII, $\mathbb{U} \subset (K \cup K')$. Then In Axiom VI replace K by $K \cup K'$ to get

$$K \cup K' \subset \mathbb{U} \tag{1}$$

Then in Definition I replace K by $K \cup K'$ and L by $\mathbb{U}$ to get

$$[K \cup K' = \mathbb{U}] \leftrightarrow [\mathbb{U} \subset (K \cup K') \wedge (K \cup K') \subset \mathbb{U}] \tag{2}$$

Then the right hand side is true by (1) and Axiom VIII, so the left hand side is true as well. □

(j). **Theorem XX:** $K \cap K' = \emptyset$

Proof. By Axiom VII, replace K by $K \cap K'$ to get

$$\emptyset \subset (K \cap K') \tag{1}$$

Then in Definition I replace K by $K \cap K'$ and L by $\emptyset$ to get

$$[K \cap K' = \emptyset] \leftrightarrow [[(K \cap K') \subset \emptyset] \wedge [\emptyset \subset (K \cap K')]] \tag{2}$$

Then the right hand side of (2) is true by (1) and Axiom IX, so the left hand side is true as well. □

Exercise 3.

This relation is identical with the relation of being intersecting. If disjointness is denoted by the symbol $\mathbb{D}$, then we might define the relation as so:

$$K\mathbb{D}L \leftrightarrow K \cap L = \emptyset$$

Exercise 4.

(arithmetic): Let $S, \cong$ be replaced by $\mathbb{N}, =$. Then Axiom I is satisfied, since $\forall x \in \mathbb{N}$, $x = x$ is true. Axiom II is satisfied since $\forall x, y, z \in \mathbb{N}$, $x = y \wedge y = z \rightarrow x = z$. Thus this is a model.

(geometry): Let $S, \cong$ be replaced by $\mathbb{T}, Z$, where $\mathbb{T}$ is the set of all triangles and Z is the relation of similarity (aZb means a is similar to b). Then Axiom I is satisfied since $\forall t \in \mathbb{T}$, tZt. Similarly, Axiom II is satisfied since $\forall t, v, w \in \mathbb{T}$, $tZv \wedge vZw \rightarrow tZw$. Then this is a model as well.

The set of all numbers with the relation $<$ is *not* a model of this axiom system: it violates Axiom I since for any number x, $x < x$ is false. Similarly, parallelism of lines *is* a model since it satisfies both axioms: if s is a straight line, then $s \parallel s$, and if a, b, c are straight lines with $a \parallel b \wedge b \parallel c$ then $a \parallel c$ as well.

Exercise 5.

In this sentence, the definiendum is the expression "x is shorter than y" or "$x < y$. The definiens is the expression "if $x \in S$, $y \in S$ and if there exists $z \in S$ such that $z \subset y$, $z \neq y$ and $x \cong z$". The expressions "$x \in S$", "$y \in S$", "$z \subset y$" and so on all belong to the calculus of classes. The symbols "$\neq$" and "$\cong$" belong to the theory of relations and geometry, respectively. The words in which these expressions are embedded belongs to the domain of sentential calculus. This definition does comply with the principles of Sections 36 and 11, since the terms to be defined are only expressed in previously understood terms of geometry, or of preceding theories (logic, etc.) and the definiendum and definiens are well formatted.

Exercise 6.

No, since Tarski does omit the formula $z \cong z$ from the hypothesis of a sentence obtained from Axiom II by a rule other than those of Section 15. However it could easily be formatted into one after laying down this rule, since it otherwise only uses the rule of detachment and the rule of substitution.

Exercise 7.

In general, to show equipoillence between systems of sentences, we have to show that each system can be derived from the other.

(a). **The system of Axiom I, Theorems I, II**

Proof.

(old from new): To prove this, we must prove Axioms I and II follow from Axiom I and Theorems I and II. Axiom I clearly follows from Axiom I, so all that remains is to show Axiom II.

From Theorem II, we have

$$x \cong y \wedge y \cong z \rightarrow x \cong z \tag{1}$$

Then by Theorem I, $y \cong z \rightarrow z \cong y$ as well. Also from by replacing y by z and vice versa in Theorem I, we also have if $z \cong y \rightarrow y \cong z$. Then, we have

$$y \cong z \leftrightarrow z \cong y \tag{2}$$

By (2), we can then replace $y \cong z$ in (1) by $z \cong y$ to get

$$x \cong y \wedge z \cong y \rightarrow x \cong z \tag{3}$$

Then, substitute z for y and y for z in (3) to obtain Axiom II.

(new from old): Theorems I and II already follow from Axioms I and II as per Section 39, so I will not re-print the proofs here.

Thus, these systems are equipollent. □

(b). **The system of Axiom I, Theorem III**

Proof.

(old from new): Since Axiom follows from itself, what remains is to prove Axiom II from Theorem III and Axiom I. From Theorem III, replace x by z to get

$$(z \cong y) \wedge (z \cong z) \rightarrow (y \cong z) \tag{1}$$

By Axiom I we have $z \cong z$ so we can simplify (1) to

$$(z \cong y) \to (y \cong z) \tag{2}$$

Then by (2) we can replace $x \cong y$ with $y \cong x$ and $x \cong z$ with $z \cong x$ in Theorem III to get

$$(y \cong x) \wedge (z \cong x) \to (y \cong x) \tag{3}$$

Then in (3) substitute y by x, z by y and x by z to get Axiom II.

(new from old): We'll use Theorem I since it has already been shown to follow from Axioms I and II. By Theorem I, in Axiom II replace $x \cong z$ by $z \cong x$ and $y \cong z$ by $z \cong y$ to get

$$(z \cong x) \wedge (z \cong y \to x \cong y) \tag{4}$$

Then in (4) replace z by x, y by z and x by y to get Theorem III.

□

(c). The system of Axiom I, Theorem IV

Proof.

(old from new): Since Axiom follows from itself, what remains is to prove Axiom II from Theorem IV and Axiom I. In Theorem IV, replace y by z to get

$$x \cong z \wedge z \cong z \to z \cong x \tag{1}$$

Then by Axiom I, $z \cong z$ so we can simplify this to

$$x \cong z \to z \cong x \tag{2}$$

Then by (2), replace $y \cong z$ by $z \cong y$ and $z \cong x$ by $x \cong z$ in Theorem IV to get

$$x \cong y \wedge z \cong y \to x \cong z \tag{3}$$

Then in (3) replace y by z and vice versa to get Axiom II.

(new from old): As before, we'll assume Theorem I, since it can also be demonstrated from Axioms I and II. By Theorem I, replace $x \cong y$ by $y \cong x$ and $y \cong z$ by $z \cong y$ in Axiom II to get

$$x \cong z \wedge z \cong y \to y \cong x \tag{4}$$

Then in (4) replace z by y and y by z to get Theorem IV.

□

(d). **The system of Axiom I, Theorems I, V**

Proof.

(old from new): Since Axiom I follows from itself, all that remains is to prove Axiom II from Theorems I, V and Axiom I. In Theorem V, replace t with z to get

$$x \cong y \wedge y \cong z \wedge z \cong z \to x \cong z \tag{1}$$

Then By Axiom I $z \cong z$, so we omit this from the hypothesis to get

$$x \cong y \wedge y \cong z \to x \cong z \tag{2}$$

Then by Theorem I, we can replace $y \cong z$ by $z \cong y$ to get

$$x \cong y \wedge z \cong y \to x \cong z \tag{3}$$

Then we can replace y by z and vice versa in (3) to get Axiom II.

(new from old): As before, we'll assume Theorem I and Theorem II, since they can be demonstrated from Axioms I and II alone. Then suppose we have x, y, z, t with $x \cong y \wedge y \cong z \wedge z \cong t$. By Theorem II, we have

$$x \cong y \wedge y \cong z \to x \cong z \tag{4}$$

Applying it again we get

$$y \cong z \wedge z \cong t \to y \cong t \tag{5}$$

Then we have

$$x \cong y \wedge y \cong z \wedge z \cong t \to x \cong y \wedge y \cong t \tag{6}$$

Then we detach $x \cong y \land y \cong t$, and by Theorem II once more we arrive at

$$x \cong y \land y \cong t \to x \cong t \tag{7}$$

and so we have

$$x \cong y \land y \cong z \land z \cong t \to x \cong y \land y \cong t \to x \cong t \tag{8}$$

which can be condensed to

$$x \cong y \land y \cong z \land z \cong t \to x \cong t \tag{9}$$

Then this is Theorem V.

$\square$

Exercise 8.

(a). For a relation R to be reflexive and to have property P, it is equivalent that the relation be symmetric, transitive, and reflexive.

(b). For a relation R to be reflexive and to have property P, it is equivalent that the relation be reflexive and have the property of Theorem IV.

(c). For a relation R to be reflexive and to have property P, it is equivalent that the relation be reflexive and (reverse?) transitive.

(d). For a relation R to be reflexive and to have property P, it is equivalent that the relation be reflexive, symmetric and have extended transitivity.

Exercise 9.

(a). Let K be the class of sets, and R be the relation of "having a non empty intersection": for $a, b \in K$, $aRb \leftrightarrow a \cap b \neq \emptyset$. Then for $a, b \in K$, aRa, and $aRb \to bRa$. However, it is not the case that if $a \leq b \land b \leq c$ then $a \leq c$.

(b). Let K be $\mathbb{Z}$ and R be $\leq$. Then for $x, y \in \mathbb{Z}$, $x \leq x$ and if $x \leq y \land y \leq z$ then $x \leq z$. However it is not the case that if $x \leq y$ then $y \leq x$.

(c). Let K be the class of humans and R be the relation of being siblings. Then for $x, y, z \in K$, $\neg(xRx)$ but if xRy then yRx. Also, if xRy and yRz then xRz.

Exercise 10.

In a certain sense, they may be unjustified in the sense of the preservation of knowledge - if indeed these theorems (taken as axioms) follow from a smaller set of axioms and vice versa, then this expanded axiom set is equipollent to the smaller one. Then, there is nothing "lost" but perhaps the proofs of these theorems. However, it is certainly desireable from the angle of being as instructive as possible to choose a minimal axiom set, so that students might observe the proofs of these theorems.

If it is the case that the systems are *not* equipollent, however, then the objection is certainly justified, since one is learning a "different" geometry than they had perhaps anticipated.

Exercise 11.

This construction of sentential calculus shares the most essential qualities of a more usual construction of deductive theories, those being the formalization of definitions and of proofs. Certainly the given rules of definition and proof are precise enough such that one is able to construct complete proofs of new sentences.

However, some distinctions are lost, such as the one between primitive and defined terms: even the most basic sentence, p, or another basic sentence $p \wedge q$ is constructed with reference to the same symbols "T" and "F" (which are *not* part of the sentential calculus, and thus cannot themselves be considered primitive terms of the theory). Similarly, the distinction between Axiom and Theorem is lost here - in order to accept something as true, it must have all "T"s in the last column of its truth table. But this method does not differ in the case of Axioms - if an Axiom is meant to be something universally accepted as true, then it should certainly be "defined" to have all Ts in its last column.

Exercise 12.

(a). $p \Delta q$

p	q	$p \Delta q$
T	T	F
F	T	F
T	F	F
F	F	T

(b). $\neg p \leftrightarrow (p\Delta p)$

p	$\neg p$	$p\Delta p$	$\neg p \leftrightarrow (p\Delta p)$
T	F	F	T
F	T	T	T

(c). $p \vee q \leftrightarrow [(p\Delta q)\Delta(p\Delta q)]$

p	q	$p \vee q$	$p\Delta q$	$(p\Delta q)\Delta(p\Delta q)$	$p \vee q \leftrightarrow [(p\Delta q)\Delta(p\Delta q)]$
T	T	T	F	T	T
F	T	T	F	T	T
T	F	T	F	T	T
F	F	F	T	F	T

(d). $p \rightarrow q \leftrightarrow [[(p\Delta p)\Delta q]\Delta[(p\Delta p)\Delta q]]$

p	q	$p \rightarrow q$	$p\Delta q$	$(p\Delta p)\Delta q$	$[(p\Delta p)\Delta q]\Delta[(p\Delta p)\Delta q]$	$p \rightarrow q \leftrightarrow [[(p\Delta p)\Delta q]\Delta[(p\Delta p)\Delta q]]$
T	T	T	F	F	T	T
F	T	T	T	F	T	T
T	F	F	F	T	F	T
F	F	T	T	F	T	T

Exercise 13.

(a). **Theorem I:** $p \rightarrow p$

Proof.

1. By Axiom I, substitute q by p to obtain:

$$p \rightarrow (p \rightarrow p) \tag{1}$$

2. By Axiom II, substitute q by p to obtain:

$$[p \rightarrow (p \rightarrow p)] \rightarrow (p \rightarrow p) \tag{2}$$

3. Apply Modus Ponens to (2) and detatch

$$p \rightarrow p \tag{3}$$

□

(b). **Theorem II:** $p \to [(p \to q) \to [(p \to q) \to q]]$

Proof.

1. In Axiom I, substitute q by $(p \to q)$ to obtain:

$$p \to [(p \to q) \to p] \tag{1}$$

2. In Axiom III, substitute q by p, r by q, and p by $(p \to q)$ to obtain:

$$[(p \to q) \to p] \to [(p \to q) \to [(p \to q) \to q]] \tag{2}$$

3. In Axiom III, substitute q by $[(p \to q) \to p]$ and r by $[(p \to q) \to [(p \to q) \to q]]$ to obtain:

$$\begin{gathered}[p \to [(p \to q) \to p]] \to \\ [[[(p \to q) \to p] \to [(p \to q) \to ((p \to q) \to q)]] \to \\ [p \to [(p \to q) \to [(p \to q) \to q]]]]\end{gathered} \tag{3}$$

4. The antecedent of (3) is true by (1), so by *modus ponens*, detatch the consequent:

$$\begin{gathered}[[(p \to q) \to p] \to [(p \to q) \to ((p \to q) \to q)]] \to \\ [p \to [(p \to q) \to [(p \to q) \to q]]]\end{gathered} \tag{4}$$

5. The antecedent of (4) is true by (2), so by *modus ponens* detatch the consequent:

$$p \to [(p \to q) \to [(p \to q) \to q]] \tag{5}$$

□

(c). **Theorem III:** $p \to [(p \to q) \to q]$

Proof.

1. In Axiom II, replace p by $(p \to q)$ to get

$$[(p \to q) \to [(p \to q) \to q]] \to [(p \to q) \to q] \tag{1}$$

2. In Axiom III, substitute q by $[(p \to q) \to [(p \to q) \to q]]$ and r by $[(p \to q) \to q]$ to obtain:

$$\begin{gathered}[p \to [(p \to q) \to [(p \to q) \to q]]] \to \\ [[[(p \to q) \to [(p \to q) \to q]] \to [(p \to q) \to q]] \to \\ [p \to [(p \to q) \to q]]]\end{gathered} \tag{2}$$

3. By Theorem II, the antecedent of (2) is true, so apply *modus ponens* to obtain the consequent:

$$\begin{gathered}[[(p \to q) \to [(p \to q) \to q]] \to [(p \to q) \to q]] \to \\ [p \to [(p \to q) \to q]]\end{gathered} \tag{3}$$

4. By (1), the antecedent of (3) is true, so apply *modus ponens* to obtain the consequent:

$$p \to [(p \to q) \to q] \tag{4}$$

□

(d). **Theorem IV:** $[p \to (q \to r)] \to [q \to (p \to r)]$

Proof.

1. In Axiom III substitute q by $q \to r$ to obtain:

$$[p \to (q \to r)] \to [[(q \to r) \to r] \to (p \to r)] \tag{1}$$

2. Again in Axiom III, substitute p by q, q by $[(q \to r) \to r]$ and r by $(p \to r)$ to get:

$$\begin{gathered}[q \to [(q \to r) \to r]] \to \\ [[[(q \to r) \to r] \to (p \to r)] \to [q \to (p \to r)]]\end{gathered} \tag{2}$$

3. In Theorem III, substitute p by q, q by r to get:

$$q \to [(q \to r) \to r] \tag{3}$$

4. By (3), the antecedent to (2) is true, so by *modus ponens* detatch:

$$[[(q \to r) \to r] \to (p \to r)] \to [q \to (p \to r)] \tag{4}$$

5. Then, by Axiom III, substitute p by $p \to (q \to r)$, q by $[[(q \to r) \to r] \to (p \to r)]$ and r by $[(q \to r) \to r]$ to obtain:

$$\begin{gathered}[[p \to (q \to r)] \to [[(q \to r) \to r] \to (p \to r)]] \to \\ [[[[(q \to r) \to r] \to (p \to r)] \to [(q \to r) \to r]] \to \\ [[p \to (q \to r)] \to [(q \to r) \to r]]]\end{gathered} \tag{5}$$

6. By (1), the antecedent of (5) is true, so by *modus ponens* detatch the consequent:

$$\begin{gathered}[[[(q \to r) \to r] \to (p \to r)] \to [(q \to r) \to r]] \to \\ [[p \to (q \to r)] \to [(q \to r) \to r]]\end{gathered} \tag{6}$$

7. By (2), the antecedent of (6) is true, so by *modus ponens* detatch the consequent:

$$[p \to (q \to r)] \to [(q \to r) \to r] \tag{7}$$

□

(e). **Theorem V:** $(\neg p) \to (p \to q)$

Proof.

1. In Axiom I, replace p by $\neg p$ and q by $\neg q$ to get:

$$\neg p \to (\neg q \to \neg p) \tag{1}$$

2. In Axiom III replace p by $(\neg p)$, q by $(\neg q \to \neg p)$ and r by $(p \to q)$ to get

$$\begin{gathered}[(\neg p) \to (\neg q \to \neg p)] \to \\ [[(\neg q \to \neg p) \to (p \to q)] \to [(\neg p) \to (p \to q)]]\end{gathered} \tag{2}$$

3. Then, the antecedent of (2) is true by (1), so by *modus ponens* detatch:

$$[(\neg q \to \neg p) \to (p \to q)] \to [(\neg p) \to (p \to q)] \tag{3}$$

4. Then, the antecendent of (3) is true by Axiom VII, so by *modus ponens* detatch:

$$(\neg p) \to (p \to q) \tag{4}$$

□

(f). **Theorem VI:** $p \to [(\neg p) \to q]$

Proof.

1. In Theorem IV, replace p by $(\neg p)$, q by p, and r by q to get:

$$[(\neg p) \to (p \to q)] \to [p \to ((\neg p) \to q)] \tag{1}$$

2. By Theorem V, the antecedent to (1) is true, so by *modus ponens* detatch:

$$p \to [(\neg p) \to q] \tag{2}$$

□

(g). **Theorem VII:** $\neg(\neg p) \to (q \to p)$

Proof.

1. In Theorem V, replace p by $\neg p$ and q by $\neg q$ to get:

$$\neg(\neg p) \to (\neg p \to \neg q) \tag{1}$$

2. In Axiom VII replace p by q and q by p to get:

$$(\neg p \to \neg q) \to (q \to p) \tag{2}$$

3. In Axiom III replace p by $\neg(\neg p)$, q by $(\neg p \to \neg q)$ and r by $(q \to p)$ to get:

$$\begin{gathered}[\neg(\neg p) \to (\neg p \to \neg q)] \to \\ [[(\neg p \to \neg q) \to (q \to p)] \to [\neg(\neg p) \to (q \to p)]]\end{gathered} \tag{3}$$

4. By (1) the antecedent to (3) is true, so by *modus ponens*, detatch:

$$[(\neg p \to \neg q) \to (q \to p)] \to [\neg(\neg p) \to (q \to p)] \tag{4}$$

5. By (2) the antecedent of (4) is true, so by *modus ponens* detatch:

$$\neg(\neg p) \to (q \to p) \tag{5}$$

□

(h). **Theorem VIII:** $\neg(\neg p) \to p$

Proof.

1. In Theorem IV, replace p by $\neg(\neg p)$ and r by p to get:

$$[\neg(\neg p) \to (q \to p)] \to [q \to (\neg(\neg p) \to p)] \tag{1}$$

2. By Theorem VII, antecedent to (1) is true, so detatch:

$$q \to (\neg(\neg p) \to p) \tag{2}$$

3. Then into (2), substitute q by $p \to p$ to get:

$$(p \to p) \to (\neg(\neg p) \to p) \tag{3}$$

4. By Axiom I, antecedent to (3) is true, so detatch:

$$\neg(\neg p) \to p \tag{4}$$

□

(i). **Theorem IX:** $p \to \neg(\neg p)$

Proof.

1. In Axiom VII substitute q by $\neg(\neg p)$ to get:

$$(\neg(\neg(\neg p)) \to \neg p) \to (p \to \neg(\neg p)) \tag{1}$$

2. In Theorem VIII substitute p by $\neg p$ to get:

$$\neg(\neg(\neg p)) \to \neg p \tag{2}$$

3. By (2), the antecedent to (1) is true, so by *modus ponens* detatch:

$$p \to \neg(\neg p) \tag{3}$$

$\square$

(j). **Theorem X:** $\neg(\neg p) \leftrightarrow p$

Proof.

1. In Axiom VI substitute p by $\neg(\neg p)$ and q by p to get:

$$(\neg(\neg p) \to p) \to [(p \to \neg(\neg p)) \to (\neg(\neg p) \leftrightarrow p)] \tag{1}$$

2. By Theorem VIII the antecedent to (1) is true, so by *modus ponens* detatch:

$$(p \to \neg(\neg p)) \to (\neg(\neg p) \leftrightarrow p) \tag{2}$$

3. By Theorem IX the antecedent to (2) is true, so by *modus ponens* detatch:

$$\neg(\neg p) \leftrightarrow p \tag{3}$$

$\square$

Exercise 14.

(a). **Theorem XI:** $(\neg p \to q) \to (p \vee q)$

Proof.

1. In Axiom V, substitute p by $p \vee q$ and q by $\neg p \to q$ to get:

$$[(p \vee q) \leftrightarrow (\neg p \to q)] \to [(\neg p \to q) \to (p \vee q)] \tag{1}$$

2. By Definition I, the antecedent of (1) is true, so by *modus ponens* detatch:

$$(\neg p \to q) \to (p \vee q) \tag{2}$$

$\square$

(b). **Theorem XII:** $p \vee \neg p$

Proof.

1. In Theorem XI, substitute q by $\neg p$ to get:

$$(\neg p \to \neg p) \to (p \vee \neg p) \tag{1}$$

2. In Theorem I, substitute p by $\neg p$ to get:

$$\neg p \to \neg p \tag{2}$$

3. By (2), antecedent of (1) is true, so by *modus ponens* detatch:

$$p \vee \neg p \tag{3}$$

□

(c). **Theorem XIII:** $p \to (p \vee q)$

Proof.

1. In Axiom III substitute q by $\neg p \to q$ and r by $p \vee q$ to get:

$$[p \to (\neg p \to q)] \to [[(\neg p \to q) \to (p \vee q)] \to [p \to (p \vee q)]] \tag{1}$$

2. By Theorem VI, the antecedent of (1) is true, so by *modus ponens* detatch:

$$[(\neg p \to q) \to (p \vee q)] \to [p \to (p \vee q)] \tag{2}$$

3. By Theorem XI, the antecedent of (2) is true, so by *modus ponens* detatch:

$$p \to (p \vee q) \tag{3}$$

□

(d). **Theorem XIV:** $p \wedge q \to \neg(\neg p \vee \neg q)$

Proof.

1. In Axiom IV, substitute p by $p \wedge q$ and q by $\neg(\neg p \vee \neg q)$ to get:

$$[p \wedge q \leftrightarrow \neg(\neg p \vee \neg q)] \to [p \wedge q \to \neg(\neg p \vee \neg q)] \tag{1}$$

2. By Definition II, the antecedent of (1) is true, so by *modus ponens* detatch:

$$p \wedge q \to \neg(\neg p \vee \neg q) \tag{2}$$

□

(e). **Theorem XV:** $\neg(\neg(p \wedge q)) \to \neg(\neg p \vee \neg q)$

Proof.

1. In Theorem VIII substitute p by $p \wedge q$ to get:

$$\neg(\neg(p \wedge q)) \to (p \wedge q) \tag{1}$$

2. Then, in Axiom III substitute p by $\neg(\neg(p \wedge q))$, q by $p \wedge q$ and r by $(\neg(\neg p \vee \neg q))$ to get:

$$\begin{gathered}[\neg(\neg(p \wedge q)) \to p \wedge q] \to \\ [[p \wedge q \to (\neg(\neg p \vee \neg q))] \to [\neg(\neg(p \wedge q)) \to (\neg(\neg p \vee \neg q))]]\end{gathered} \tag{2}$$

3. Then, by (1), the antecedent of (2) is true, so by *modus ponens* detatch:

$$[p \wedge q \to (\neg(\neg p \vee \neg q))] \to [\neg(\neg(p \wedge q)) \to (\neg(\neg p \vee \neg q))] \tag{3}$$

4. By Theorem XIV, the antecedent of (3) is true, so by *modus ponens* detatch:

$$\neg(\neg(p \wedge q)) \to \neg(\neg p \vee \neg q) \tag{4}$$

□

(f). **Theorem XVI:** $(\neg p \vee \neg q) \to \neg(p \wedge q)$

Proof.

1. In Axiom VII, substitute p by $(\neg p \vee \neg q)$ and q by $\neg(p \wedge q)$ to get:

$$[\neg(\neg(p \wedge q)) \rightarrow \neg(\neg p \vee \neg q)] \rightarrow [(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)] \tag{1}$$

2. By Theorem XV, the antecedent of (1) is true, so by *modus ponens* detatch:

$$(\neg p \vee \neg q) \rightarrow \neg(p \wedge q) \tag{2}$$

$\square$

(g). **Theorem XVII:** $\neg p \rightarrow \neg(p \wedge q)$

Proof.

1. In Theorem XIII substitute p by $\neg p$ and q by $\neg q$ to get:

$$\neg p \rightarrow \neg p \vee \neg q \tag{1}$$

2. In Axiom III substitute p by $\neg p$, q by $\neg p \vee \neg q$ and r by $\neg(p \wedge q)$ to get:

$$[\neg p \rightarrow \neg p \vee \neg q] \rightarrow [[\neg p \vee \neg q \rightarrow \neg(p \wedge q)] \rightarrow [\neg p \rightarrow \neg(p \wedge q)]] \tag{2}$$

3. By (1), the antecedent to (2) is true, so by *modus ponens* detatch:

$$[\neg p \vee \neg q \rightarrow \neg(p \wedge q)] \rightarrow [\neg p \rightarrow \neg(p \wedge q)] \tag{3}$$

4. By Theorem XVI, the antecedent to (3) is true, so by *modus ponens* detatch:

$$\neg p \rightarrow \neg(p \wedge q) \tag{4}$$

$\square$

(h). **Theorem XVIII:** $p \wedge q \rightarrow p$

Proof.

1. In Axiom VII substitute p by $p \wedge q$ and q by p to get:

$$(\neg(p) \rightarrow \neg(p \wedge q)) \rightarrow (p \wedge q \rightarrow p) \tag{1}$$

2. By Theorem XVII, the antecedent to (1) is true, so by *modus ponens* detatch:

$$p \wedge q \rightarrow p \tag{2}$$

□

Exercise 15.

Definition 1. $p \Delta q \leftrightarrow \neg p \wedge \neg q$

Exercise 16.

(a). **Axiom I:** $p \rightarrow (q \rightarrow p)$

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
F	T	F	T
T	F	T	T
F	F	T	T

(b). **Axiom II:** $[p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)$

p	q	$p \rightarrow q$	$p \rightarrow (p \rightarrow q)$	$[p \rightarrow (p \rightarrow q)] \rightarrow (p \rightarrow q)$
T	T	T	T	T
F	T	T	T	T
T	F	F	F	T
F	F	T	T	T

(c). **Axiom III:** $(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(q \to r) \to (p \to r)$	$(p \to q) \to [(q \to r) \to (p \to r)]$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	T	T
F	T	F	T	F	T	T	T
T	F	F	F	T	F	F	T
F	F	F	T	T	T	T	T

(d). **Axiom IV:** $(p \leftrightarrow q) \to (p \to q)$

p	q	$p \leftrightarrow q$	$p \to q$	$(p \leftrightarrow q) \to (p \to q)$
T	T	T	T	T
F	T	F	T	T
T	F	F	F	T
F	F	T	T	T

(e). **Axiom V:** $(p \leftrightarrow q) \to (q \to p)$

p	q	$p \leftrightarrow q$	$q \to p$	$(p \leftrightarrow q) \to (q \to p)$
T	T	T	T	T
F	T	F	F	T
T	F	F	T	T
F	F	T	T	T

(f). **Axiom VI:** $(p \to q) \to [(q \to p) \to (p \leftrightarrow q)]$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(q \to p) \to (p \leftrightarrow q)$	$(p \to q) \to [(q \to p) \to (p \leftrightarrow q)]$
T	T	T	T	T	T	T
F	T	T	F	F	T	T
T	F	F	T	F	F	T
F	F	T	T	T	T	T

(g). **Axiom VII:** $[\neg(q) \to \neg(p)] \to (p \to q)$

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$	$[\neg(q) \to \neg(p)] \to (p \to q)$
T	T	F	F	T	T	T
F	T	T	F	T	T	T
T	F	F	T	F	F	T
F	F	T	T	T	T	T

(h). **Definition I:** $(p \vee q) \leftrightarrow [(\neg) \to q]$

p	q	$\neg p$	$p \vee q$	$\neg p \to q$	$(p \vee q) \leftrightarrow [(\neg) \to q]$
T	T	F	T	T	T
F	T	T	T	T	T
T	F	F	T	T	T
F	F	T	F	F	T

(i). **Definition II:** $(p \wedge q) \leftrightarrow \neg[\neg p \vee \neg q]$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \vee \neg q$	$\neg(\neg p \vee \neg q)$	$(p \wedge q) \leftrightarrow \neg[\neg p \vee \neg q]$
T	T	F	F	T	F	T	T
F	T	T	F	F	T	F	T
T	F	F	T	F	T	F	T
F	F	T	T	F	T	F	T

(j). **Definition 15.I:** $p\Delta q \leftrightarrow \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p\Delta q$	$p\Delta q \leftrightarrow \neg p \wedge \neg q$
T	T	F	F	F	F	T
F	T	T	F	F	F	T
T	F	F	T	F	F	T
F	F	T	T	T	T	T

Exercise 17.

The method of truth tables provides a simple solution of the decision problem for the sentential calculus: for any true statement of the sentitial calculus, its truth table will return a final column of all Ts. Therefore, for any true sentence S, we can construct the truth table for its contradictory sentence $\neg S$, which will consequently have all Fs in its last column. Then, it is clear by this method that no two contradictory sentences can be true. Thus, sentential calculus is consistent, so we have positive solution to its decision problem.

Exercise 18.

We'll refer to the following law of sentential calculus as Law A: *For any p and q, if p and not p, then q.*

Suppose S is an inconsistent axiom system of a deductive theory presupposing sentential calculus. Consider any sentence b formulated in the terms of S. Then, S is inconsistent, so there exists a sentence $a \in S$ with $a \wedge \neg a$ true. Then by the law of replacement, in Law A substitute p by a and q by b to get the true sentence $a \wedge \neg a \to b$. Then, the antecedent $a \wedge \neg a$ is true as well, so b must be true by *modus ponens*.

Exercise 19.

If a deductive theory is complete, then all true sentences of the theory are provable. Thus, any sentences that can be formulated but not proved must be false, since if they were true, they would be provable.

Thus, extending the theory by adding all formulatable but unprovable sentences would be to add false sentences to the theory. Take any of these false sentences now considered as part of the theory. If a sentence is false, its negation is true - but if it's negation is true, it was already in the system (by completeness). Then, the system contains (at least) this pair of contradictory sentences, so it is not consistent.

Exercise 20.

The terms ”negation”, ”contradictory sentences”, ”conjunction”, ”disjunction”, ”implication”, ”antecedent”, ”consequent”, ”equivalence”, ”hypothesis”, ”conclusion”, ”definition”, ”definiendum”, ”definiens”, ”inverse”, ”converse”, ”contrapositive”, ”conjugate”, ”rule of proof”, ”rule of definition”, and ”complete proof” all belong to the field of the methodology of deductive sciences.